1 Black holes

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Abstract. This paper is concerned with several not-quantum aspects of black holes, with emphasis on theoretical and mathematical issues related to numerical modeling of black hole space-times. Part of the material has a review character, but some new results or proposals are also presented. We review the experimental evidence for existence of black holes. We propose a definition of black hole region for any theory governed by a symmetric hyperbolic system of equations. Our definition reproduces the usual one for gravity, and leads to the one associated with the Unruh metric in the case of Euler equations. We review the global conditions which have been used in the Scri-based definition of a black hole and point out the deficiencies of the Scri approach. Various results on the structure of horizons and apparent horizons are presented, and a new proof of semi-convexity of horizons based on a variational principle is given. Recent results on the classification of stationary singularity-free vacuum solutions are reviewed. Two new frameworks for discussing black holes are proposed: a "naive approach", based on coordinate systems, and a "quasi-local approach", based on timelike boundaries satisfying a null convexity condition. Some properties of the resulting black holes are established, including an area theorem, topology theorems, and an approximation theorem for the location of the horizon.

1.1 Introduction

Black holes belong to the most fascinating objects predicted by Einstein's theory of gravitation. Although they have been studied for years,[1] they still attract quite a lot of attention in the physics and astrophysics literature and, in fact, an exponential growth of the number of related papers can be observed.[2] It turns out that several field theories are known to possess solutions which exhibit black hole properties:

- The "standard" gravitational ones which, according to our current postulates, are black holes for all classical fields.
- The "dumb holes", which are the sonic counterparts of black holes, first discussed by Unruh [127].

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[1] The reader is referred to the introduction to [27] for an excellent concise review of the history of the concept of a black hole, and to [26, 82] for more detailed ones.

[2] A search on the key word "black holes" on 2.XI.2001 reveals 213 papers on gr-qc for the current year and 773 papers from 1991, date of the beginning of the archive; the figures on astro-ph and hep-th are very similar.

- The "optical" ones – the black-hole counterparts arising in the theory of moving dielectric media, or in non-linear electrodynamics [95, 111].
- The "numerical black holes" – objects constructed by numerical general relativists. We shall argue below that this leads to the need of introducing, and studying, new frameworks for the notion of black holes; two such frameworks ("naive black holes" and "quasi-local black holes") will be introduced and studied in Sections 1.8.1 and 1.8.2 below.

(An even longer list of models and submodels can be found in [9].) In this work we shall discuss various aspects of the above. The reader is referred to [21, 51, 77, 83, 113, 133] and references therein for a review of quantum aspects of black holes. Let us start with a short review of the observational status of black holes in astrophysics.

1.2 Experimental evidence

While there is growing evidence that black holes do indeed exist in astrophysical objects, and that alternative explanations for the observations discussed below seem less convincing, it should be borne in mind that no undisputed evidence of occurrence of black holes has been presented so far. The flagship black hole candidate used to be Cygnus X-1, known and studied for years (*cf., e.g.,* [27]). Its published mass has been going up and down over time: an optimistic interpretation of this phenomenon is that there has been mass accretion during the ascending periods, and Hawking radiation during the descending ones; a more realistic one is that there is still considerable uncertainty in the determination of this mass. Table 1.1[3] lists a series of very strong black hole candidates in X-ray binary systems; M_c is mass of the compact object and M_* is that of its optical companion; some other candidates, as well as references, can be found in [103, 109]. The binaries have been divided into two families: the High Mass X-ray Binaries (HMXB), where the companion star is of (relatively) high mass, and the Low Mass X-ray Binaries (LMXB), where the companion is typically below a solar mass. The LMXB's include the "X-ray transients", so-called because of flaring-up behaviour. This particularity allows to make detailed studies of their optical properties during the quiescent periods, which would be impossible during the periods of intense X-ray activity. The stellar systems listed have X-ray spectra which are neither periodic (that would correspond to a rotating neutron star), nor recurrent (which is interpreted as thermonuclear explosions on a neutron star's hard surface). The final selection criterion is that of the mass M_c exceeding the Chandrasekhar limit $M_C \approx 3$ solar masses $M_\odot$.[4] According to the authors of [27], the strongest black hole candidate in 1999 was V404 Cygni, which belongs to the LMXB class.

Table 1.1 should be put into perspective by realizing that, by some estimates [99], a typical galaxy – such as ours – should harbour $10^7 - 10^8$ stellar

[3] The recent review [109] lists thirteen black hole candidates.

[4] See [103] for a discussion and references concerning the value of M_C.

Table 1.1. Stellar mass black hole candidates (from [99])

Type	Binary system	$M_c/M_\odot$	$M_*/M_\odot$
HMXB:	Cygnus X-1	11–21	24–42
	LMC X-3	5.6 –7.8	20
	LMC X-1	≥ 4	4–8
LMXB:	V 404 Cyg	10–15	≈ 0.6
	A 0620-00	5–17	0.2–0.7
	GS 1124-68 (Nova Musc)	4.2–6.5	0.5–0.8
	GS 2000+25 (Nova Vul 88)	6-14	≈ 0.7
	GRO J 1655-40	4.5 – 6.5	≈ 1.2
	H 1705-25 (Nova Oph 77)	5–9	≈ 0.4
	J 04224+32	6–14	$\approx 0.3 - 0.6$

black mass holes. We note an interesting proposal, put forward in [28], to carry out observations by gravitational microlensing of some 20 000 stellar mass black holes that are predicted [105] to cluster within 0.7 pc of Sgr A* (the centre of our galaxy).

It is now widely accepted that quasars and active galactic nuclei are powered by accretion onto massive black holes [101, 135]. Further, over the last few years there has been increasing evidence that massive dark objects may reside at the centres of most, if not all, galaxies [100, 120]. In several cases the best explanation for the nature of those objects is that they are "heavyweight" black holes, with masses ranging from 10^6 to 10^{10} solar masses. Table 1.2[5] lists some supermassive black hole candidates; some other candidates, as well as precise references, can be found in [87, 103, 104, 119]. The main criterion for finding candidates for such black holes is the presence of a large mass within a small region; this is determined by maser line spectroscopy, gas spectroscopy, or by measuring the motion of stars orbiting around the galactic nucleus. The reader is referred to [108] for a discussion of the maser emission lines and their analysis for the supermassive black hole candidate NGC 4258. An example of measurements via gas spectrography is given by the analysis of the Hubble Space Telescope (HST) observations of the radio galaxy M 87 [126] (compare [101]): A spectral analysis shows the presence of a disk-like structure of ionized gas in the innermost few arc seconds in the vicinity of the nucleus of M 87. The velocity of the gas measured by spectroscopy (cf. Fig. 1.1) at a distance from the nucleus of the order of 6×10^{17} m, shows that the gas recedes from us on one side, and approaches us on

[5] The table lists those galaxies which are listed both in [104] and [87]; we note that some candidates from earlier lists [119] do not occur any more in [87, 104]. Nineteen of the observations listed have been published in 2000 or 2001.

Table 1.2. Twenty-nine supermassive black hole candidates (from [87, 104])

dynamics of	host galaxy	$M_h/M_\odot$	host galaxy	$M_h/M_\odot$
water maser discs:	NGC 4258	4×10^7		
gas discs:	IC 1459	2×10^8	M 87	3×10^9
	NGC 2787	4×10^7	NGC 3245	2×10^8
	NGC 4261	5×10^8	NGC 4374	4×10^8
	NGC 5128	2×10^8	NGC 6251	6×10^8
	NGC 7052	3×10^8		
stars:	NGC 821	4×10^7	NGC 1023	4×10^7
	NGC 2778	1×10^7	NGC 3115	1×10^9
	NGC 3377	1×10^8	NGC 3379	1×10^8
	NGC 3384	1×10^7	NGC 3608	1×10^8
	NGC 4291	2×10^8	NGC 4342	3×10^8
	NGC 4473	1×10^8	NGC 4486B	5×10^8
	NGC 4564	6×10^7	NGC 4649	2×10^9
	NGC 4697	2×10^8	NGC 4742	1×10^7
	NGC 5845	3×10^8	NGC 7457	4×10^6
	Milky Way	2.5×10^6		

the other, with a velocity difference of about 920 km s^{-1} . This leads to a mass of the central object of $\sim 3 \times 10^9\ M_\odot$, and no form of matter can occupy such a small region except for a black hole. Figure 1.2 shows another image, reconstructed out of HST observations, of a recent candidate for a supermassive black hole – the (active) galactic nucleus of NGC 4438 [85]. To close the discussion of Table 1.2, we note that the determination of mass of the galactic nuclei via direct measurements of star motions has been made possible by the unprecedentedly high angular resolution and sensitivity of the HST. There seems to be consensus [87, 104, 120] that the two most convincing supermassive black hole candidates are the galactic nuclei of NGC 4258 and of our own Milky Way.

There is a huge gap between the masses of stellar mass black holes and those of the supermassive black holes and, in Bromley's terminology [20], "middleweight" candidates would be welcome. Such a tentative black hole with M of the order of $460M_\odot$ has been identified in M 82 by A. Ptak and R. Griffiths [117]. E. Colbert and R. Mushotzky [48] give a list of compact X-ray sources which could include some further black holes with masses in the $10^2 - 10^4 M_\odot$ range.

Let us close this section by pointing out the review paper [25] which discusses both theoretical and experimental issues concerning *primordial* black holes.

1.3 Causality for symmetric hyperbolic systems

The usual, Scri based, definition of a black hole proceeds as follows: one starts by introducing a preferred asymptotic region – this is usually taken to be a conformal boundary. Then the black hole region is defined as the set of points which cannot send information to the asymptotic region; we will return to the details of this construction in Section 1.4 below. The key ingredient here is the notion of "not being able to send information" – this is usually defined using the metric, together with the postulate that the propagation speed for objects carrying information is bounded from above by that of null geodesics. The aim of this section is to provide a precise mathematical version of this notion, via Proposition 1 below, for any symmetric hyperbolic system of differential equations, without introducing any supplementary postulates. The point is that every physical theory defines its own causality via the corresponding system of equations. We will then show in Section 1.3.1 how the "Unruh metric" arises in the application of our construction to the Euler equations.

Let, thus, L be a quasi-linear, symmetric hyperbolic, first order partial differential operator acting on sections φ of a bundle E, with scalar product $\langle \cdot, \cdot \rangle$,

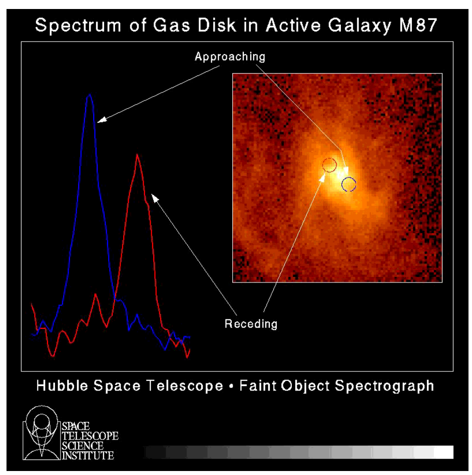

Fig. 1.1. Hubble Space Telescope observations of spectra of gas in the vicinity of the nucleus of the radio galaxy M 87, NASA and H. Ford (STScI/JHU) [123].

over a manifold $\mathscr{M}$; in a local trivialization over a coordinate patch $\mathscr{U} \ni x^\mu$ we have

$$L[\varphi] := A^\mu(\varphi, x^\beta)\partial_\mu\varphi + B(\varphi) \;, \tag{1.1}$$

where the A^μ's are endomorphisms of the fibers of E, symmetric with respect to $\langle\cdot,\cdot\rangle$. Given a section φ of E (defined perhaps on a subset of $\mathscr{M}$), a hypersurface $\mathscr{S}$ is said to be *spacelike*[6] if one of its fields of co-normals n_μ satisfies

$$A^\mu\left(\varphi\left(p\right), p\right) n_\mu(p) > 0 \;, \quad p \in \mathscr{S} \;;$$

this is understood in the sense of strict positive definiteness of endomorphisms. By definition of a symmetric hyperbolic system, every point of the domain of definition of φ lies on some spacelike hypersurface. For $p \in \mathscr{M}$ we set

$$\mathscr{T}^\flat_p := \{\alpha \in T^*_p\mathscr{M} \mid A^\mu\left(\varphi\left(p\right), p\right)\alpha_\mu \text{ is positive definite}\} \subset T^*_p\mathscr{M} \;, \tag{1.2a}$$

$$\mathscr{C}^\flat_p := \overline{\mathscr{T}^\flat_p} \subset \{\alpha \in T^*_p\mathscr{M} \mid A^\mu\left(\varphi\left(p\right), p\right)\alpha_\mu \text{ is non-negative definite}\} \subset T^*_p\mathscr{M} \;, \tag{1.2b}$$

and

$$\mathscr{T}^\flat := \cup_{p\in\mathscr{M}}\mathscr{T}^\flat_p \subset T^*\mathscr{M} \;, \qquad \mathscr{C}^\flat := \cup_{p\in\mathscr{M}}\mathscr{C}^\flat_p = \overline{\mathscr{T}^\flat} \;. \tag{1.3}$$

The bundle $\mathscr{T}^\flat$ is thus the bundle of covectors normal to spacelike hypersurfaces. Because positive definite matrices form a convex set, each set $\mathscr{T}^\flat_p$ is a convex

[6] In quasi-linear theories, in which the coefficients of the highest derivatives depend upon the fields, the notion of spacelikeness will of course depend upon the given set of fields.

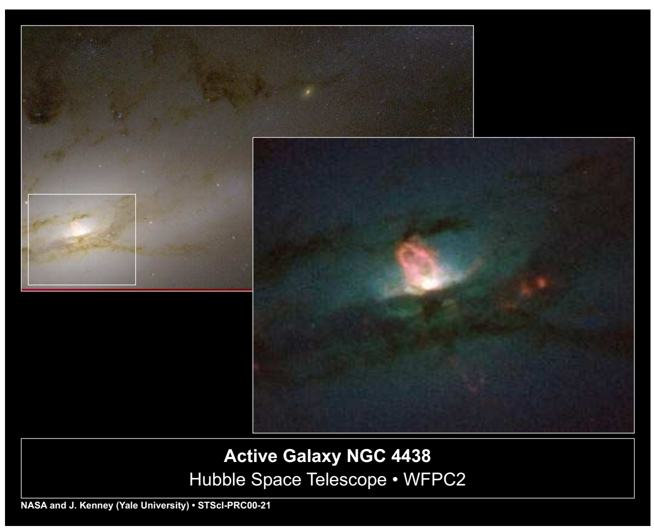

Fig. 1.2. Hubble Space Telescope observations [85] of the nucleus of the galaxy NGC 4438, from the STScI Public Archive [123].

cone in $T^*_p\mathscr{M}$. The bundles $\mathscr{T}^\sharp \subset T\mathscr{M}$, respectively $\mathscr{C}^\sharp \subset T\mathscr{M}$, of *timelike*, respectively *causal*, vectors are defined by duality:

$$\mathscr{T}^\sharp_p := \{X \in T_p\mathscr{M} \; |\forall\, \alpha \in \mathscr{T}^\flat_p \;\; X^\mu\alpha_\mu > 0\} \;, \tag{1.4a}$$

$$\mathscr{C}^\sharp_p := \overline{\mathscr{T}^\sharp_p} = \{X \in T_p\mathscr{M} \; |\forall\, \alpha \in \mathscr{T}^\flat_p \;\; X^\mu\alpha_\mu \geq 0\} \;. \tag{1.4b}$$

Recall that all standard classical field equations, i.e., Euler's equations, the scalar wave equation, the wave map equation, Maxwell's equations, Yang-Mills equations, or Einstein's equations, can be written as a symmetric hyperbolic system (cf., e.g., [118] or [125, Vol. IV]). For example, the wave equation for a scalar field u on a Lorentzian manifold $(\mathscr{M}, g)$ can be written in the form (1.1) by introducing a new set of variables

$$\varphi := (u, e_\mu(u)) \;,$$

where the e_μ's, $\mu = 0, \ldots, n$, form an ON-frame for g. In this case $\mathscr{T}^\sharp$ coincides with the bundle of timelike future pointing vectors of the metric g (where the future direction is determined by that of e_0), $\mathscr{C}^\sharp$ coincides with the bundle of causal future pointing or vanishing vectors of g, while $\mathscr{T}^\flat$ and $\mathscr{C}^\flat$ are, modulo sign (we are using the signature $(-, +, \ldots, +)$), "$\mathscr{T}^\sharp$ and $\mathscr{C}^\sharp$ with indices lowered using the metric". The same remains true for wave maps, for linear electrodynamics (considered as a first order system for $\boldsymbol{E}$ and $\boldsymbol{B}$), for the Yang-Mills equations in the Lorentz gauge, and for the usual symmetric hyperbolic reduction of Einstein's equations based on harmonic coordinates. (See the end of this section for more comments about the Einstein case.)

We emphasize that the bundles $\mathscr{T}^\sharp$, etc., are defined solely by the system of equations under consideration, independently of any metric. They can be used to determine the causality properties of a given symmetric hyperbolic system by mimicking the usual metric constructions [13, 73, 112, 115]: *timelike future directed paths* are defined as piecewise differentiable maps γ from I to $\mathscr{M}$, where $I \subset \mathbb{R}$ is an interval, with the property that the tangent $\dot\gamma$ is in $\mathscr{T}^\sharp$ wherever defined; *causal future directed paths* are defined as Lipschitz continuous paths such that $\dot\gamma$ is in $\mathscr{C}^\sharp$ wherever defined. *Past directed paths* are paths which are obtained from future directed paths by a reversal of parameterization. The *timelike future* $I^+(\Omega)$, respectively the *causal future* $J^+(\Omega)$, of a set $\Omega \subset \mathscr{M}$ is defined as the set of points which can be reached form Ω by following a future directed timelike, respectively causal, path. Causal and timelike futures J^- and I^- are obtained form the above definition by replacing "future" by "past". Notions such as global hyperbolicity, and so on, can be defined in the usual way. It might be convenient to write $I^\pm(\Omega, L)$, etc., to emphasize the dependence upon the system of equations L, if ambiguities can arise. We note that for semi-linear systems – by definition, these are the systems in which the coefficients A^μ in (1.1) do not depend upon φ – the resulting causal constructs are independent of the solution φ under consideration; however, this will not be the case in general.

We are ready now to address the issue of main interest in this work – the notion of a black hole. Given a set $\Omega \subset \mathscr{M}$ and a symmetric hyperbolic system

L we define the *black hole region* $\mathscr{B}_\Omega(L)$ *of* Ω as

$$\mathscr{B}_\Omega(L) := \mathscr{M} \setminus J^-(\Omega, L) \ . \tag{1.5}$$

The object so defined depends upon the set Ω, and acquires its full meaning when Ω is naturally distinguished by the problem at hand.

In several cases, some of which have already been mentioned, the causality theory constructed above arises out of a metric; we will show in Section 1.3.1 below that this holds also for the Euler equations. The intuitive meaning associated to the notion of a black hole region is that no information from $\mathscr{B}_\Omega(L)$ can reach Ω. This is made precise by the following result, the proof of which proceeds by causality arguments which are known in principle:

Proposition 1 *Let* φ *be a solution of*

$$L[\varphi] = J \ , \tag{1.6}$$

where L *is symmetric hyperbolic operator (1.1) on a manifold* $\mathscr{M}$*, and suppose that there exists a Lorentzian metric* g *such that the sets* $\mathscr{C}^\sharp$ *defined in (1.4b) are future light cones of* g*. Assume that* $(\mathscr{M}, g)$ *is globally hyperbolic with Cauchy surface* $\mathscr{S}$ *and consider a set* $\Omega \subset \mathscr{M}$ *such that* $\mathscr{B}_\Omega(L) \neq \emptyset$*. If* φ' *is a solution of*

$$L[\varphi'] = J' \quad \text{with} \quad \varphi|_{\mathscr{S}} = \varphi'|_{\mathscr{S}} \ , \tag{1.7}$$

and if the difference of the source terms $J - J'$ *is supported in* $\mathscr{B}_\Omega(L)$*, then* $\varphi = \varphi'$ *on* $J^-(\Omega, L)$*.*

We believe that the conclusion above is true *without* the assumption that causality is determined by a metric, but we are not aware of any studies of this question in the current context; it would be of interest to settle this. The reader is referred to [97] for an analogous analysis of strictly hyperbolic systems, and to [30] for that of second order Lagrangean hyperbolic systems.

We close this section by mentioning that care has to be taken when interpreting the notions above for systems of equations which are not directly in first order symmetric hyperbolic form: it can happen that the physical equations can be then rewritten in several different symmetric hyperbolic forms, leading to different notions of pasts, futures, black hole regions, etc. This does indeed happen for the Einstein equations: in [57, Section 4.1] a symmetric hyperbolic system is presented which can be used to solve the Einstein equations and in which the causal cones, as defined above, differ from the metric ones; see the discussion after Equation (4.25) there. Another example of this kind arises if the T-vector of the system of equations given in Sect. 5.2 of [55] is chosen to be spacelike. This is related to the acausal (in the usual metric sense) propagation of gauge degrees of freedom in the associated systems, and does of course not affect the propagation of any physically relevant quantities. Given a system of equations, say (S), a simple way out of this problem is to define the physical black hole region as the intersection, as L runs over the set of those symmetric hyperbolic systems L the solutions of which include solutions of (S), of the associated black hole regions $\mathscr{B}_\Omega(L)$. We note that the existence of exact solutions such as the pp-waves, where nonlinear perturbations propagate along null geodesics, guarantees that every symmetric hyperbolic system L which reproduces the Einstein's equations will necessary contain the metric light cone as its causal light cone associated to L. It follows that the above prescription does indeed reproduce the usual metric definition of a black hole region for the vacuum Einstein equations.

1.3.1 Dumb holes

Let us illustrate the considerations of the previous section in the case of the Euler equations

$$\begin{aligned} \rho\left(\dot{\mathbf{v}} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla p(\rho) = 0\ , \\ \dot{\rho} + \nabla(\rho\mathbf{v}) = 0\ . \end{aligned} \tag{1.8}$$

For strictly positive ρ's (1.8) can be rewritten as a symmetric hyperbolic system of the form (1.1) with $B = 0$, for a field $\varphi = (v^i, \rho)$, by introducing the matrices

$$A^0 = \begin{pmatrix} \rho\,\mathrm{id}_{\mathbb{R}^3} & 0 \\ 0 & \frac{p'}{\rho} \end{pmatrix}\ , \quad A^i = \begin{pmatrix} \rho v^i \mathrm{id}_{\mathbb{R}^3} & p'\delta^i_j \\ p'\delta^i_j & \frac{p'}{\rho}v^i \end{pmatrix}\ , \tag{1.9}$$

which are clearly symmetric with respect to the standard scalar product on $\mathbb{R}^3 \times \mathbb{R} \ni (\boldsymbol{v}, \rho)$. Straightforward algebra shows that $A^\mu n_\mu$ is strictly positive if and only if $\rho > 0$, $dp/d\rho > 0$ and

$$-(n_0 + v^i n_i)^2 + p' \sum_i (n_i)^2 < 0 \quad \Longleftrightarrow \quad g^{\mu\nu} n_\mu n_\nu < 0\ , \tag{1.10}$$

where the *Unruh metric* g is defined as [127, 130]

$$g^{\mu\nu} = \frac{1}{c_s \rho}\begin{pmatrix} -1 & -v^i \\ -v^j & c_s^2\delta^{ij} - v^i v^j \end{pmatrix} \Longleftrightarrow g_{\mu\nu} = \frac{\rho}{c_s}\begin{pmatrix} -c_s^2 + \boldsymbol{v}^2 & -v^i \\ -v^j & \delta_{ij} \end{pmatrix}\ , \tag{1.11}$$

$$c_s^2 = \frac{dp}{d\rho}\ .$$

This metric exhibits a typical black hole behaviour for stationary solutions in which the speed $\boldsymbol{v}$ of the fluid velocity meets the speed of sound c_s across a smooth hypersurface.

The original argument of Unruh leading to (1.11) was a perturbational one, for irrotational flows:

$$\boldsymbol{v} = \nabla\Phi\ . \tag{1.12}$$

Small perturbations $\phi \equiv \delta\Phi$ of Φ satisfy then a scalar wave equation [127],

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0\ ,$$

in the metric (1.11). Our discussion above shows that neither the irrotationality of the flow, nor the perturbative character of the argument are essential for the problem at hand. The fact that causality for the full non-linear Euler's equations is governed by a metric can essentially be found in Courant and Hilbert, and is certainly well known to some researchers (H. Friedrich, private communication).

There was renewed interest in the above because of the recent experiments with the Bose-Einstein condensates. Recall that in the Madelung formulation, the Gross-Pitaevskii equation governing the dynamics of the condensates can be rewritten in Euler form. There have been suggestions that some of the effects associated to black-hole type causality, including the analogue of the Hawking radiation, could be observed in such systems, see [10, 83, 131] and references therein. While the resulting black holes are still referred to as *sonic*, this is rather misleading as the underlying Gross-Pitaevskii equation does not have anything to do with sound propagation in liquid or gaseous physical media.

1.3.2 Optical holes

According to M. Visser [128], in a dielectric fluid "the refractive index n, the fluid velocity v, and the background Minkowski metric η can be combined algebraically to provide an effective metric g"

$$g_{\mu\nu} = \eta_{\mu\nu} - (n^2 - 1)v_\mu v_\nu \; .$$

This leads to black-hole effective geometries for refractive indices which exceed 1 in some regions. Phenomenological models for this behaviour have been proposed by U. Leonhardt and P. Piwnicki, cf. [95] and references therein[7]. Similarly, a (different) effective metric is obtained in nonlinear electrodynamics, leading – according to [111] – to models which sometimes contain closed timelike curves. The status of these "analogous models" seems to be somewhat less clear than that of the sonic ones.

1.3.3 Trapped surfaces

We shall close this section by noting the discussion in [96, 129] concerning the notion of trapped surfaces[8] for black hole geometries – such surfaces are useful detectors of black hole regions in GR with matter fields carying positive energy. It should be borne in mind that while causality concepts can be introduced for any symmetric hyperbolic system, the existence of a trapped surface signals the presence of a black hole region *only* when causality is governed by *a metric satisfying an energy positivity condition* (and, if in a Scri context, when Scri satisfies various regularity conditions, see Section 1.4 below). In the non-gravitational black hole models the metric is the one arising out of the causality structure of the theory, and there are no reasons in general to believe that its Ricci tensor should have any preferred properties; to start with, it is defined only up to a conformal factor anyway. In addition, the "trapped surface" terminology is used

[7] See also URL `http://www.st-and.ac.uk/~www_pa/group/quantumoptics`

[8] Cf. Section 1.6 for the differential-geometric definition of this notion

in [129] for objects which are completely unrelated to the usual differential geometric context, which is extremely misleading and obscures the problems at hand.

1.4 Standard black holes

The standard way of defining black holes is by using conformal completions: A pair $(\widetilde{\mathscr{M}}, \tilde{g})$ is called a *conformal completion* of $(\mathscr{M}, g)$ if $\widetilde{\mathscr{M}}$ is a manifold with boundary such that:

1. $\mathscr{M}$ is the interior of $\widetilde{\mathscr{M}}$,
2. there exists a function Ω, with the property that the metric $\tilde{g}$, defined to be $\Omega^2 g$ on $\mathscr{M}$, extends by continuity to the boundary of $\widetilde{\mathscr{M}}$, with the extended metric still being non-degenerate throughout,
3. Ω is positive on $\mathscr{M}$, differentiable on $\widetilde{\mathscr{M}}$, vanishes on $\mathscr{I}$, with $d\Omega$ *nowhere vanishing* on $\mathscr{I}$.

We emphasize that no assumptions about the causal nature of Scri are made so far. The boundary of $\widetilde{\mathscr{M}}$ will be called Scri, denoted $\mathscr{I}$. In the usual textbooks [73, 132] smoothness of both the conformal completion and the metric $\tilde{g}$ is imposed, though this can be weakened for many purposes. We set

$$\mathscr{I}^+ = \{p \in \mathscr{I} \mid I^-(p; \widetilde{\mathscr{M}}) \cap \mathscr{M} \neq \emptyset\} .$$

Assuming various global regularity conditions on $\widetilde{\mathscr{M}}$, to which we shall return in Sect. 1.4.1, one then defines the black hole region $\mathscr{B}$ as

$$\mathscr{B} := \mathscr{M} \setminus J^-(\mathscr{I}^+) . \tag{1.13}$$

Let us point out some drawbacks of this approach:

- **Non-equivalent Scri's:** Conformal completions at null infinity do not have to be unique, an example can be constructed as follows: the Taub–NUT metrics can be locally written in the form [106]

$$-U^{-1}dt^2 + (2L)^2 U\sigma_1^2 + (t^2 + L^2)(\sigma_2^2 + \sigma_3^2) , \tag{1.14}$$

$$U(t) = -1 + \tfrac{2(mt+L^2)}{t^2+L^2} . \tag{1.15}$$

  where σ_1, σ_2 and σ_3 are left invariant one–forms on $SU(2) \approx S^3$. The constants L and m are real numbers with $L > 0$. Parameterizing S^3 with Euler angles (μ, θ, φ) one is led to the following form of the metric

$$g = -U^{-1}dt^2 + (2L)^2 U(d\mu + \cos\theta d\varphi)^2 + (t^2 + L^2)(d\theta^2 + \sin^2\theta d\varphi^2) .$$

  The standard way of performing extensions across the Cauchy horizons $t_\pm := M \pm \sqrt{M^2 + L^2}$ is to introduce new coordinates

$$(t, \mu, \theta, \varphi) \to (t, \mu \pm \int_{t_0}^{t} [2LU(s)]^{-1} ds, \theta, \varphi) , \tag{1.16}$$

which gives

$$g_\pm = \pm 4L(d\mu + \cos\theta d\varphi)dt \\ +(2L)^2 U(d\mu + \cos\theta d\varphi)^2 + (t^2 + L^2)(d\theta^2 + \sin^2\theta d\varphi^2)\,. \quad (1.17)$$

Each of the metrics $g_\pm$ can be smoothly conformally extended to the boundary at infinity "$t = \infty$" by introducing

$$x = 1/t\,,$$

so that (1.17) becomes

$$g_\pm = x^{-2}\Big(\mp 4L(d\mu + \cos\theta d\varphi)dx \\ +(2L)^2 x^2 U(d\mu + \cos\theta d\varphi)^2 + (1 + L^2x^2)(d\theta^2 + \sin^2\theta d\varphi^2)\Big) \quad (1.18)$$

In each case this leads to a Scri diffeomorphic to S^3. There is a simple isometry between g_+ and g_- given by

$$(x, \mu, \theta, \varphi) \to (x, -\mu, \theta, -\varphi)$$

(this does correspond to a smooth map of the region $t \in (t_+, \infty)$ into itself, cf. [42]), so that the two Scri's so obtained are isometric. However, in addition to the two ways of attaching Scri to the region $t \in (t_+, \infty)$ there are the two corresponding ways of extending this region across the Cauchy horizon $t = t_+$, leading to four possible manifolds with boundary. It can then be seen, using e.g. the arguments of [42], that the four possible manifolds split into two pairs, each of the manifolds from one pair *not* being isometric to one from the other. Taking into account the corresponding completion at "$t = -\infty$", and the two extensions across the Cauchy horizon $t = t_-$, one is led to four inequivalent conformal completions of each of the two inequivalent [42] time-oriented, maximally extended, standard Taub-NUT space-times.

Under mild completeness conditions in the spirit of [66], uniqueness of $\mathscr{I}^+$ as a point set in *past-distinguishing* space-times should follow from the TIP and TIF construction of [67]; however, uniqueness of Scri's differentiable structure within that framework is far from being clear.

Yet another approach to uniqueness has been proposed by Geroch [65]: A completion as defined at the beginning of this section is said to be *a regular asymptote* if, given any point $p \in \mathscr{I}$, and any non-zero null vector $\ell \in T_q\widetilde{\mathscr{M}}$ such that the maximally extended null geodesic with tangent ℓ does not meet p, then there are neighbourhoods $\mathscr{U} \subset \widetilde{\mathscr{M}}$ of p and $\mathscr{V} \subset T\widetilde{\mathscr{M}}$ of ℓ such that no maximally extended null geodesic with tangent in $\mathscr{V}$ meets $\mathscr{U}$. This regularity condition is not satisfied by the Taub-NUT completions described above. Supposing that the set of regular asymptotes is non-empty, Geroch [65, Theorem 2, p. 14] gives an argument for existence of a maximal regular asymptote, unique up to well behaved conformal transformations.

Now, we have not been able to fill in the details[9] of some of the arguments suggested in [65], but the construction given is plausible enough so that we expect that it might actually be correct. Whatever the case, Geroch's regularity requirement is a global condition which seems to be difficult to control in general situations.[10] In particular it could happen that many space-times of interest admitting conformal completions do not admit any regular ones. It should be borne in mind that one of the main applications of the $\mathscr{I}$ framework is the possibility of *uniquely* defining global charges such as mass, angular momentum, etc. (cf. [43] for a new approach to this question), in which issues about global behaviour of *all* geodesics seem completely irrelevant. The fact that there is no suitable uniqueness result for the differentiable structure of conformal completions is a serious gap in our understanding of objects such as the Trautman-Bondi mass.

We note that uniqueness of a class of Riemannian conformal completions at infinity has been established in [41, Section 6]; this result can probably be used to obtain uniqueness of differentiable structure of Lorentzian conformal completions for Scri's admitting cross-sections. Further partial results on the problem at hand can be found in [121].

- **Poorly differentiable Scri's:** In all standard treatments [65, 73, 132] it is assumed that both the conformal completion $\widetilde{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}$ and the extended metric $\tilde{g}$ are smooth, or have a high degree of differentiability [114]. This is a restriction which excludes most space-times which are asymptotically Minkowskian in lightlike directions, see [5, 90] and references therein. Poor differentiability properties of $\mathscr{I}$ change the peeling properties of the gravitational field [44], but most – if not all – essential properties of black holes should be unaffected by conformal completions with, e.g., polyhomogeneous differentiability properties as considered in [6, 44]. It should, however, be borne in mind that the hypothesis of smoothness has been done in the standard treatments, so that in a complete theory the validity of various claims should be reexamined. Some new results concerning existence of space-times with a poorly differentiable Scri can be found in [94].
- **The structure of i^+:** The current theory of black holes is entirely based on intuitions originating in the Kerr and Schwarzschild geometries. In those space-times we have a family of preferred "stationary" observers which follow the orbits of the Killing vector field ∂_t in the asymptotic region, and their past coincides with that of $\mathscr{I}^+$. It is customary to denote by i^+ the set con-

[9] It is not completely clear that the map φ defined by Geroch in his proof is simultaneously differentiable for all regular asymptotes, as asserted in [65, p. 14]. The regularity condition guarantees Hausdorffness of the maximal regular asymptote constructed in [65, p. 14], but does not seem to guarantee its differentiability in any obvious way.

[10] Globally hyperbolic conformal completions (in the sense of manifolds with boundary) should satisfy Geroch's condition. Nevertheless, it should be borne in mind that good causal properties of a space-time might fail to survive the process of adding a conformal boundary. For example, adding a Cauchy horizon to a maximal globally hyperbolic space-time sometimes leads to space-times which are not globally hyperbolic in the sense of manifolds with boundary.

sisting of the points $t = \infty$, where t is the Killing time parameter for those observers. Now, the usual conformal diagrams for those space-times [73, 107] leave the highly misleading impression that i^+ is a regular point in the conformally rescaled manifold, which, to the best of our knowledge, is not the case. In dynamical cases the situation is likely to become worse. For example, one can imagine black hole space-times with a conformal diagram which, to the future of a Cauchy hypersurface $t = 0$, looks as in Fig. 1.3. In that

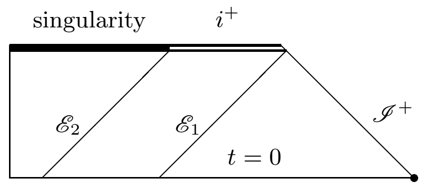

Fig. 1.3. An asymptotically flat space-time with an unusual i^+.

diagram the set i^+ should be thought of as the addition to the space-time manifold $\mathscr{M}$ of a set of points "$\{t = \infty, q \in \mathscr{O}\}$", where $t \in [0, \infty)$ is the proper time for a family of observers $\mathscr{O}$. The part of the boundary of $\widetilde{\mathscr{M}}$ corresponding to i^+ is a singularity of the conformally rescaled metric, but we assume that it does *not* correspond to singular behaviour in the physical space-time. In this space-time there is the usual event horizon $\mathscr{E}_1$ corresponding to the boundary of the past of $\mathscr{I}^+$, which is completely irrelevant for the family of observers $\mathscr{O}$, and an event horizon $\mathscr{E}_2$ which is the boundary of the *true* black hole region for the family $\mathscr{O}$, i.e., the region that is not accessible to observations for the family $\mathscr{O}$. Clearly the usual black hole definition carries no physically interesting information in such a setting.

- **Causal regularity of Scri:** As already pointed out, in order to be able to prove interesting results the definition (1.13) should be complemented by causal conditions on $\widetilde{\mathscr{M}}$. The various approaches to this question, discussed in Sect. 1.4.1, are aesthetically highly unsatisfactory: it appears reasonable to impose causal regularity conditions on a space-time, but why should some unphysical completion have any such properties? Clearly, the physical properties of a black hole should not depend upon the causal regularity – or lack thereof – of some artificial boundary which is being attached to the space-time. While it seems reasonable and justified to restrict attention to space-times which possess good causal properties, it is not clear why the addition of artificial boundaries should preserve those properties, or even be consistent with them. Physically motivated restrictions are relevant when dealing with physical objects, they are not when non-physical constructs are considered.

- **Inadequacy for numerical purposes:** Several[11] numerical studies of black holes have been performed on numerical grids which cover finite space-time regions [1, 2, 19, 86]. Clearly, it would be convenient to have a set-up which is more compatible in spirit with such calculations than the Scri one.

We will suggest, in Sections 1.8.1 and 1.8.2 below, two approaches in which the above listed problems are avoided. Before doing this, let us complete the presentation of the usual, Scri based, approach to black holes.

1.4.1 Scri regularity conditions, and the area theorem

It is easily seen that the definition (1.13) is not very useful without some *completeness conditions* on $\mathscr{I}^+$. For instance, denote by $\widetilde{\mathscr{M}}$ the standard conformal completion[12] of the Minkowski space-time $\mathbb{R}^{1,3}$, with $\mathscr{I}^+ \approx \mathbb{R} \times S^2$, where the $\mathbb{R}$ factor is parameterized by a Bondi coordinate u, with $u = t - r$ on $\mathbb{R}^{1,3}$. One can obtain a new completion $\widetilde{\mathscr{M}}_2$ by replacing $\mathscr{I}^+$ by an open subset thereof, e.g. restricting the range of u's to $(-\infty, 0)$. This will give a non-empty black hole region $\mathscr{B}$ in $\mathbb{R}^{1,3}$,

$$\mathscr{B} = J^+(0) ,$$

which is clearly a completely uninteresting statement. A way out of this problem (as well as of some other related ones, discussed in [66]) has been proposed[13] in [66], for space-times which are vacuum near the conformal boundary, assuming that the cosmological constant is zero: R. Geroch and G. Horowitz introduce a preferred family of conformal factors Ω such that the matrix HessΩ of second covariant derivatives of Ω vanishes at $\mathscr{I}$. It is then required that both $\mathscr{I}^+$ and $\mathscr{I}^-$ be diffeomorphic to $\mathbb{R} \times S^2$, with the $\mathbb{R}$ factor corresponding to integral curves of $\widetilde{\nabla}\Omega$, which are assumed to be complete.

Let us pass to a presentation of the causal regularity conditions which are imposed in Hawking and Ellis's approach [73]. Now, a set of conditions is only useful insofar as it allows to prove something. Since the "area theorem"[14] is one of the landmark theorems in the theory of black holes, we shall concentrate on the set of conditions from Hawking and Ellis as used in their treatment of the area theorem; we follow the presentation of [38, Appendix B]. One of the conditions of the Hawking–Ellis area theorem [73, Proposition 9.2.7, p. 318] is that spacetime $(\mathscr{M}, g)$ is *weakly asymptotically simple and empty* ("WASE", [73, p. 225]). This means that there exists an open set $\mathscr{U} \subset \mathscr{M}$ which is isometric

[11] The numerical simulations in [12, 16, 70] cover regions extending all the way to infinity, within frameworks which seem to be closely related to the "naive" framework of Section 1.8.1 below.

[12] Inroduce coordinates $u = t - r$, $x = 1/r$, θ, φ on $R^4 \setminus \{r = 0\}$, with (r, θ, φ) – the usual spherical coordinates on $\mathbb{R}^3$, and set $\mathscr{I}^+ := \{x = 0\}$ in those coordinates.

[13] The "WASE setup" of [73], presented below, does not solve the problem, cf. [66].

[14] Recall that the area theorem is the statement that the area of cross-sections of event horizons is non-increasing towards the future. One possible precise version of this theorem can be found in Theorem 8 below.

to $\mathscr{U}' \cap \mathscr{M}'$, where $\mathscr{U}'$ is a neighborhood of null infinity in an asymptotically simple and empty (ASE) spacetime $(\mathscr{M}', g')$ [73, p. 222]. It is further assumed that $\mathscr{M}$ admits a partial Cauchy surface $\mathscr{S}$ with respect to which $\mathscr{M}$ is *future asymptotically predictable* ([73], p. 310). This is defined by the requirement that $\mathscr{I}^+$ is contained in the closure of the future domain of dependence $\mathcal{D}^+(\mathscr{S};\mathscr{M})$ of $\mathscr{S}$, where the closure is taken in the conformally completed manifold $\overline{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}^+ \cup \mathscr{I}^-$, with both $\mathscr{I}^+$ and $\mathscr{I}^-$ being null hypersurfaces. Next, one says that $(\mathscr{M}, g)$ is *strongly future asymptotically predictable* ([73], p. 313) if it is future asymptotically predictable and if $J^+(\mathscr{S}) \cap \bar{J}^-(\mathscr{I}^+;\overline{\mathscr{M}})$ is contained in $\mathcal{D}^+(\mathscr{S};\mathscr{M})$. Finally ([73], p. 318), $(\mathscr{M}, g)$ is said to be a *regular predictable space* if $(\mathscr{M}, g)$ is strongly future asymptotically predictable and if the following three conditions hold:

(α) $\mathscr{S} \cap \bar{J}^-(\mathscr{I}^+;\overline{\mathscr{M}})$ is homeomorphic to $\mathbb{R}^3 \setminus$(an open set with compact closure).
(β) $\mathscr{S}$ is simply connected.
(γ) the family of hypersurfaces $\mathscr{S}(\tau)$ constructed in [73, Proposition 9.2.3, p. 313] has the property that for sufficiently large τ the sets $\mathscr{S}(\tau) \cap \bar{J}^-(\mathscr{I}^+;\overline{\mathscr{M}})$ are contained in $\bar{J}^+(\mathscr{I}^-;\overline{\mathscr{M}})$.

It should be clear at this stage that this set of conditions is rather intricate. As it turns out, it is not clear whether or not it does suffice for a proof of the area theorem, as asserted in [73, Proposition 9.2.7, p. 318]: indeed, in the proof of [73, Proposition 9.2.1, p. 311] (which is one of the results used in the proof of [73, Proposition 9.2.7, p. 318]) Hawking and Ellis write: "This shows that if $\mathscr{W}$ is any compact set of $\mathscr{S}$, every generator of $\mathscr{I}^+$ leaves $J^+(\mathscr{W};\overline{\mathscr{M}})$." The justification of this given in [73] is wrong. If one is willing to impose the sentence in quotation marks as *yet one more regularity hypothesis* on $\mathscr{I}^+$, then the arguments given in [73] apply to prove the area theorem for black holes with a piecewise smooth event horizon; the results in [38] show that this remains true with no supplementary conditions on the differentiability of the horizon.

Not only is the above set of hypotheses aesthetically unappealing, it is far from being unique. An alternative way to guarantee that the area theorem will hold in the "future asymptotically predictable WASE" set–up of [73] is to impose some mild additional conditions on $\mathscr{U}$ and $\mathscr{S}$. There are quite a few possibilities, one such set of conditions has been described in [38, Appendix B]: Let $\psi : \mathscr{U} \to \mathscr{U}' \cap \mathscr{M}'$ denote the isometry arising in the definition of the WASE spacetime $\mathscr{M}$. First, one requires that ψ can be extended by continuity to a continuous map, still denoted by ψ, defined on $\overline{\mathscr{U}}$. Next, one demands that there exists a compact set $K \subset \mathscr{M}'$ such that,

$$\psi(J^+(\mathscr{S};\mathscr{M}) \cap \partial\mathscr{U}) \subset J^+(K;\mathscr{M}') \,, \tag{1.19}$$

see Fig. 1.4. Under those conditions the area theorem does again hold.

Several other proposals how to modify the WASE conditions of [73] to obtain sufficient control of the space–times at hand have been proposed in [47, 89, 110].

A completely different, and considerably simpler, treatment of the question of regularity of Scri needed for the area theorem, has been proposed in [38]: A point

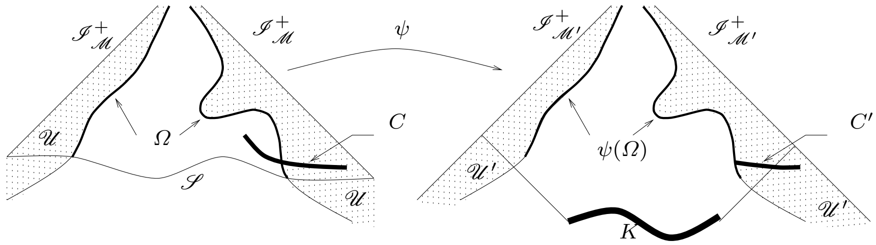

Fig. 1.4. The set $\Omega \equiv J^+(\mathscr{S};\mathscr{M}) \cap \partial\mathscr{U}$ and its image under ψ.

q in a set $A \subset B$ is said to be a *past* point of A with respect to B if $J^-(q;B) \cap A = \{q\}$. One then says that $\mathscr{I}^+$ is $\mathscr{H}$*–regular* if there exists a neighborhood $\mathcal{O}$ of $\mathscr{H}$ such that for every compact set $C \subset \mathcal{O}$ satisfying $I^+(C;\bar{M}) \cap \mathscr{I}^+ \neq \emptyset$ there exists a past point *with respect to* $\mathscr{I}^+$ in $\partial I^+(C;\bar{M}) \cap \mathscr{I}^+$. The condition of $\mathscr{H}$–regularity of $\mathscr{I}^+$ is precisely what is needed for the arguments of the area theorem to go through. This condition is somewhat related to the i^0*-avoidance condition* used by Galloway and Woolgar [63].

Yet another approach has been advocated by Wald [132]: there one considers globally hyperbolic completions in a set-up that includes i^0. While the conditions of [132] do lead to a coherent set-up for the validity of the area theorem, they introduce several new difficulties related to the low differentiability of the conformally rescaled metric at i^0, not to mention the question of mere existence of such completions. This can actually be relaxed to the related weaker requirement of global hyperbolicity of $\mathscr{M} \cup \mathscr{I}^+$ (in the sense of a manifold with boundary) which enforces $\mathscr{H}$–regularity of $\mathscr{I}^+$ [38], so that the area theorem again holds.

1.5 Horizons

A key notion related to the concept of a black hole is that of an *event horizon*,

$$\mathscr{E} := \partial\mathscr{B} \,. \tag{1.20}$$

(Actually, when $\mathscr{E}$ is not connected some care is required for a useful definition, but this does not affect the discussion that follows.) Event horizons are a special case of a family of objects called *future horizons*: by definition, these are closed topological hypersurfaces $\mathscr{H}$ threaded by null geodesics, called *generators*, with no *future end* points, and possibly with *past end* points. At the latter, differentiability of $\mathscr{H}$ breaks down in general; a necessary and sufficient condition for this breakdown has been given in [14]. It seems that most authors have been taking for granted that horizons are nice, piecewise smooth hypersurfaces. This is, however, not the case, and examples of *nowhere* C^1 horizons have been constructed in [40]. Further, nowhere C^1 horizons are generic in the class of convex horizons in Minkowski space–time [23]. This leads to various difficulties when trying to study their structure, e.g., attempting to prove results such as the area theorem discussed in Sect. 1.4.1, compare Theorem 8 below. Horizons are always *locally Lipschitz-continuous* topological hypersurfaces [115] (compare Corollary A.3 below), i.e., they are locally graphs of functions satisfying

$$|\phi(x) - \phi(y)| \leq C|x - y| \ .$$

In spite of the potential low differentiability, the usual optical scalars can be defined almost everywhere in an *Alexandrov sense*, as follows: A function is said to be *semi–convex* if it is the sum of a C^2 function and of a convex function. It has been shown in [38][15] that future horizons are semi-convex. We give an alternative proof of this result in an appendix: the new argument has some interest on its own, as it proceeds via a variational principle for horizons, which could be useful for numerical applications. The interest of the semi-convexity property relies on the following theorem of Alexandrov:

Theorem 2 (Alexandrov [52, Appendix E]) *Semi–convex functions $f : B \to \mathbb{R}$ are "twice–differentiable"* almost everywhere *in the following sense: there exists a set $B_{\mathcal{A}l}$ with full measure in B such that*

$$\begin{aligned} &\forall\ x \in B_{\mathcal{A}l}\ \exists\ Q \in (\mathbb{R}^p)^* \otimes (\mathbb{R}^p)^* \text{ such that } \forall\ y \in B \\ &\qquad f(y) - f(x) - df(x)(y-x) = Q(x-y, x-y) + r_2(x,y)\ , \end{aligned}$$

with $r_2(x,y) = o(|x-y|^2)$. The symmetric quadratic form Q above will be denoted by $\frac{1}{2}D^2 f(x)$, and will be called the second Alexandrov derivative *of f at x.*

Points in $B_{\mathcal{A}l}$ will be referred to as *Alexandrov points* of B.

One can now use Theorem 2 to define the equivalent to the usual divergence θ of the horizon, by writing $\mathscr{H}$ locally as the graph of a function f, and using the second Alexandrov derivatives of f to define (almost everywhere on $\mathscr{H}$) the divergence $\theta_{\mathcal{A}l}$ of the horizon. More precisely, let $p = (t = f(q), q)$ be an Alexandrov point of $\mathscr{H}$ and let e_i, $i = 1, 2, 3$ a basis of $T_p\mathscr{H}$ such that

$$e_3 = K_\mu(p)dx^\mu = -dt + df(q)\ ,\ g(e_1, e_1) = (e_2, e_2) = 1\ ,\ g(e_1, e_2) = 0\ .$$

One then sets

$$\nabla_i K_j = D^2_{ij} f - \Gamma^\mu_{ij} K_\mu\ , \qquad \theta_{\mathcal{A}l} = (e^i_1 e^j_1 + e^i_2 e^j_2)\nabla_i K_j\ .$$

where the Hessian matrix $D^2_{ij} f$ above is that of second Alexandrov derivatives. This reduces to the standard definition of θ when $f \in C^2$.

Several results of causality theory go through with θ replaced by $\theta_{\mathcal{A}l}$, though the standard arguments sometimes have to be replaced by completely different ones, the reader is referred to [38] for details. In particular the Raychaudhuri equation, as well as the remaining optical equations, hold on almost all generators of $\mathscr{H}$.

Somewhat surprisingly, event horizons in smooth, stationary, globally hyperbolic, asymptotically flat space-times satisfying the null energy condition are always smooth null hypersurfaces, analytic if the metric is analytic; this is a corollary of the area theorem of [38].

[15] Some further results concerning differentiability properties of horizons can be found in [14, 35, 39].

1.6 Apparent horizons

In spite of their name, apparent horizons are *not* horizons. They are usually [73] defined on spacelike hypersurfaces $\mathscr{S} \subset \mathscr{M}$, as follows: let $\Omega \subset \mathscr{M}$ be the set covered by *(future) trapped surfaces*: by definition, those are compact, boundaryless, smooth surfaces $S \subset \mathscr{S}$ with the property that

$$\theta(S) := \lambda - (g^{ij} - n^i n^j)K_{ij} < 0 \ , \tag{1.21}$$

where λ is the (outwards) mean extrinsic curvature of S in $\mathscr{S}$, n^i is the outwards pointing unit normal to S in $\mathscr{S}$, and K_{ij} is the extrinsic curvature of $\mathscr{S}$ in $\mathscr{M}$. There is no reason for Ω to be nonempty in general; on the other hand, in appropriately censored space-times, a non-empty Ω implies the existence of a black hole region. Hawking and Ellis define the apparent horizon $\mathscr{A}$ as

$$\mathscr{A} := \partial\Omega \ , \tag{1.22}$$

and argue that

$$\theta(\mathscr{A}) = 0 \ . \tag{1.23}$$

Their argument is correct if one assumes that $\mathscr{A}$ is C^2. However, $\mathscr{A}$ could be a priori a very rough set, with θ not even defined, in which case the arguments of [73] do not apply. Probably the simplest way out is to use (1.23) as the definition of apparent horizon, and forget about Ω. The existence of a C^2 compact, boundaryless surface satisfying (1.23) does again imply the existence of a black hole in a Scri framework, assuming appropriate causal properties of $\mathscr{I}^+$.

We note that if $K_{ij} = 0$, then (1.22) is the equation for a minimal surface. Now, in the course of their proof of the Penrose inequality, G. Huisken and T. Ilmanen [80] prove that the outermost minimal (in the sense of calculus of variations) surface is always smooth, which supports the validity of (1.23) at least for the outermost component of $\mathscr{A}$. However, they provide examples which show that this outermost minimal surface might not be the boundary of Ω in general. This illustrates another deficiency of the definition (1.22). Some partial results concerning the differentiability properties of $\mathscr{A}$ have been obtained by M. Kriele and S. Hayward in [88]. R. Howard and J. Fu have recently shown [79] that $\partial\Omega$ satisfies (1.22) in a *viscosity sense* – this is defined as follows: Let D be an open set, and let $p \in \partial D$. Then U is an inner (outer) support domain if U is an open subset of D (of $\complement D$), ∂U is a C^2 hypersurface and $p \in \partial U$. One says that $\theta(\partial D) \leq 0$ (respectively $\theta(\partial D) \geq 0$) in the viscosity sense if for any point $p \in \partial D$ and for any inner support (outer support) domain we have

$$\theta(\partial U)(p) \leq 0 \qquad (\text{respectively} \ \ \theta(\partial U)(p) \geq 0) \ .$$

This coincides with the usual inequality $\theta \leq 0$ ($\theta \geq 0$) on any subset of ∂D which is C^2. Finally $\theta(\partial D) = 0$ in the viscosity sense if for any point $p \in \partial D$ we have both $\theta(\partial D) \leq 0$ in the viscosity sense, and $\theta(\partial D) \geq 0$ in the viscosity sense.

It is tempting to conjecture the following: if $\widehat{\Omega}$ is the set covered by smooth compact S's satisfying

$$\theta(S) := \lambda + (g^{ij} - n^i n^j)K_{ij} \leq 0 \ ,$$

then any outermost connected component of $\partial\widehat{\Omega}$ is smooth, separating, and satisfies (1.22). When $K_{ij} = 0$ this statement is known to be true by the already mentioned result of G. Huisken and T. Ilmanen.

We refer the reader to [3, 72] and references therein for a numerical treatment of apparent horizons.

1.7 Classification of stationary solutions ("No hair theorems")

Uniqueness theorems for stationary solutions have been one of the central fields of research in the theory of black holes, the reader is referred to [33, 34] and references therein for a review, as well as lists of open problems; cf. also [75, 76]. Actually the perspective here is somewhat larger, as one is interested in classifying all solutions, not only those that contain black holes. Since the time of writing of [33, 34] some progress has been made, both for zero and non-zero cosmological constant; it is convenient to discuss various cases separately. Throughout this section, *stationary* means that there exists an action of $\mathbb{R}$ on $\mathscr{M}$ by isometries, with the orbits of the group being timelike sufficiently far away in an asymptotic region; *strictly stationary* means that the orbits are timelike everywhere. Stationary black hole solutions are never strictly stationary.

$\Lambda = 0$: The following fundamental theorem has been recently proved by M. Anderson:

Theorem 3 (Anderson [4]) *The only geodesically complete strictly stationary solutions of the vacuum Einstein equations which do not contain closed timelike curves are the Minkowski space-time and quotients thereof.*

This is a milestone result in general relativity, involving only the natural geometric conditions of completeness together with absence of closed timelike curves. This should be contrasted with the version of this result originating in Lichnerowicz's work, the state-of-the art form of which, as based on previous techniques, assumes *in addition* global hyperbolicity, *and* asymptotic flatness, *and* existence of a compact interior. Last but not least, the paper introduces techniques which have not been used in mathematical general relativity so far, and which are likely to be useful tools in future work.

In the black hole case, the only significant progress made since the review paper [34] is for *static* black holes: In [36] the classification of static vacuum black holes which contain an asymptotically flat spacelike hypersurface with compact interior has been, essentially,[16] finished. This problem has a long history, starting with the pioneering work of Israel [81]. The most complete result existing

[16] Some comments about the qualification "essentially" are in order: uniqueness theorems usually proceed in two steps, the first being the reduction of the problem to a Riemannian one, the second consisting of a uniqueness theorem for the Riemannian problem. The latter part of the proof in [36] seems to be optimal; one could consider improving the former, see [36] for a detailed discussion.

in the literature prior to [36] was that of Bunting and Masood-ul-Alam [24] who showed, roughly speaking, that all appropriately regular such black holes which *do not contain degenerate horizons* belong to the Schwarzschild family. In [36] the condition of non-degeneracy of the event horizon is removed, showing that the Schwarzschild black holes exhaust the family of all appropriately regular black hole space-times. The proof requires an extension of the positive mass theorem which applies to asymptotically flat *complete* Riemannian manifolds, proved in [11]. The paper [37] contains an improvement of similar previous results concerning the electro-vacuum black holes, under the restrictive condition that all degenerate components of the black hole carry charges of the same sign. This seems a reasonable condition from a physical point of view: opposite charges would attract, which added to the gravitational attraction should prevent the configuration from being static. However, a proper mathematical treatment which would transform this argument into a proof is still missing.

$\Lambda > 0$: The natural boundary conditions for stationary solutions with $\Lambda > 0$ are the "no boundary ones": one considers globally hyperbolic space-times containing compact, boundaryless spatial surfaces. In this case there is no asymptotic region, and *stationarity* is defined by the requirement that the set of points at which the Killing vector field is timelike is non-empty. In [92] J. Lafontaine and L. Rozoy have announced a complete classification of static space-times with a compact (boundaryless) totally geodesic spacelike hypersurface $\mathscr{S}$, under the assumption that the Lorentzian norm $\sqrt{-g_{\mu\nu}X^\mu X^\nu}$ of the Killing vector field is a Morse-Bott function, and that of analyticity[17] of the metric on $\mathscr{S}$. The hypothesis that $\mathscr{S}$ is totally geodesic is the usual hypothesis of staticity. This is the first result of such generality in this context. The proof is an extension of an argument of R. Beig and W. Simon [15] done in a related context, and proceeds by proving that the metric on $\mathscr{S}$ must be conformally flat; one can then conclude using [91].

Uniqueness results under the completely different hypothesis of existence of a conformal completion have been previously established by H. Friedrich [54] and by Boucher [17].

$\Lambda < 0$: Our understanding of the class of stationary space-times with a negative cosmological constant is much poorer than that of the $\Lambda = 0$ ones. It seems that the only results available so far concern static space-times. To start with, the question of boundary conditions which should be imposed in the asymptotic region are far from being understood, a beginning of a systematic analysis of this question can be found in [45]. A uniqueness argument for the de Sitter metric in the strictly stationary case has been proposed in [18]. It is only recently that complete proofs of the required positive energy theorem have been given [41, 134,

[17] This is a condition on the analyticity of the metric at the set of points at which the Killing vector vanishes or becomes null, since the metric is necessarily analytic elsewhere.

136]; the results in [41] do actually lead to a uniqueness result under much weaker asymptotic conditions than originally suggested [18]. It should be pointed out that the nature of the corresponding Riemannian problem is completely different from that of the asymptotically flat case: here one needs to analyze the set of solutions of the equations for a Riemannian metric g and a function V on a three dimensional manifold $\mathscr{S}$,

$$\Delta V = -\Lambda V \ , \tag{1.24}$$

$$R_{ij} = V^{-1} D_i D_j V + \Lambda g_{ij} \ , \tag{1.25}$$

and it is customary to assume that $(\mathscr{S}, g)$ can be conformally compactified. Now, the uniqueness theorems mentioned require that $V^{-2}g$ extends by continuity to a smooth metric on the conformal boundary, with constant scalar curvature there. No reasons are known why this should be the case, and it is conceivable that more general static solutions would exist which do not satisfy this condition. It would be of interest to construct such solutions, or to prove their non-existence.

The topology of the boundary of the conformally compactifiable solutions of (1.24)-(1.25) does not have to be spherical, as is the case for the anti-de Sitter metric, while the uniqueness result mentioned above assumes that the boundary at infinity $\partial_\infty \mathscr{S}$ is a sphere. When $\partial_\infty \mathscr{S}$ is a torus, a strictly static solution has been found by Horowitz and Myers [78]. The metric in $n+1 \geq 4$ spacetime dimensions is

$$ds^2 = -r^2 dt^2 + \frac{1}{V(r)} dr^2 + V(r) d\phi^2 + r^2 \sum_{i=1}^{n-2} (dy^i)^2 \ , \tag{1.26}$$

where $V(r) = \frac{r^2}{\ell^2}\left(1 - \frac{\rho^n}{r^n}\right)$, $\ell^2 = -\frac{n(n-1)}{2\Lambda}$, and ρ is a constant. Regularity demands that ϕ be identified with period $\beta_0 = \frac{4\pi\ell^2}{n\rho}$. The periods of the y^i's are arbitrary. The space-time metric is "asymptotically locally anti-de Sitter" with boundary at conformal infinity foliated by spacelike $(n-1)$-tori. The time slices of space-time itself, when conformally completed, have topology $D^2 \times T^{n-2}$, which in dimension 3+1 is a solid torus. Under a convexity condition on the conformal boundary, G. Galloway, S. Surya and E. Woolgar have proved a uniqueness theorem for the corresponding metrics [62].

The above mentioned results concern the strictly static case, and the question of classification of black holes with a Killing vector which is not timelike everywhere is essentially open. In [45] an argument has been presented which shifts the problem of proving uniqueness of a class of such black holes to that of proving a Penrose-type inequality[18] for conformally compactifiable Riemannian manifolds with minimal boundary, and with scalar curvature bounded from below by a negative constant. While there is some hope that some such results could be proved by extending the arguments of [80], this remains to be seen.

[18] A version of such an inequality has been originally proposed by Gibbons [69]; the inequality proposed there is not quite correct, see [45].

1.8 Black holes without Scri

There has been considerable progress in the numerical analysis of black hole solutions of Einstein's equations; here one of the objectives is to write a stable code which would solve the full four dimensional Einstein equations, with initial data containing a non-connected black hole region that eventually merges into a connected one. One wishes to be able to consider initial data which do not possess any symmetries, and which have various parameters – such as the masses of the individual black holes, their angular momenta, as well as distances between them – which can be varied in significant ranges. Finally one wishes the code to run to a stage where the solution settles to a state close to equilibrium. The challenge then is to calculate the gravitational wave forms for each set of parameters, which could then be used in the gravitational wave observatories to determine the parameters of the collapsing black holes out of the observations made. This program has been being undertaken for years by several groups of researchers, with steady progress being made [1, 2, 12, 16, 19, 70, 71, 86, 93].[19]

There is a fundamental difficulty above, of deciding whether or not one is dealing indeed with the desired black hole initial data: the definition (1.13) of a black hole requires a conformal boundary $\mathscr{I}$ satisfying some – if not all – properties discussed in Sect. 1.4.1. Clearly there is no way of ensuring those requirements in a calculation performed on a finite space-time grid.[20]

In practice what one does is to set up initial data on a finite grid so that the region near the boundary is close to flat (in the conformal approach the whole asymptotically flat region is covered by the numerical grid, and does not need to be near the boundary of the numerical grid; this distinction does not affect the discussion here). Then one evolves the initial data as long as the code allows. The gravitational waves emitted by the system are then extracted out of the metric near the boundary of the grid. Now, our understanding of energy emitted by gravitational radiation is essentially based on an analysis of the metric in an asymptotic region where g is nearly flat. In order to recover useful information out of the numerical data it is thus necessary for the metric near the boundary of the grid to remain close to a flat one. If we want to be sure that the information extracted contains all the essential dynamical information about the system, the metric near the boundary of the grid should quiet down to an almost stationary state as time evolves. Now, it is straightforward to set-up

[19] Some spectacular visualizations of the calculations performed can be found at the URL `http://jean-luc.aei.mpg.de/NCSA1999/GrazingBlackHoles`

[20] The conformal approach developed by Friedrich (cf., e.g., [56] and references therein, as well as S. Husa's and J. Frauendiener's contributions to this volume) provides an ideal numerical framework for studying gravitational radiation in situations where the extended space-time is smoothly conformally compactifiable across i^+, since then one can hope that the code will be able to "calculate Scri" globally to the future of the initial hyperboloidal hypersurface. It is not clear whether a conformal approach could provide more information than the non-conformal ones when i^+ is itself a singularity of the conformally rescaled equations, as is the case for black holes.

a mathematical framework to describe such situations without having to invoke conformal completions, this is done in the next section.

1.8.1 Naive black holes

Consider a *globally hyperbolic* space-time $\mathscr{M}$ which contains a region covered by coordinates (t, x^i) with ranges

$$r := \sqrt{\sum_i (x^i)^2} \geq R_0 \;, \qquad T_0 - R_0 + r \leq t < \infty \;, \tag{1.27}$$

such that the metric g satisfies there

$$|g_{\mu\nu} - \eta_{\mu\nu}| \leq C_1 r^{-\alpha} \leq C_2 \;, \quad \alpha > 0 \;, \tag{1.28}$$

for some positive constants C_1, C_2, α; clearly C_2 can be chosen to be less than or equal to $C_1 R_0^{-\alpha}$. Making R_0 larger one can thus make C_2 as small as desired, e.g.

$$C_2 = 10^{-2} \;, \tag{1.29}$$

which is a convenient number in dimension $3+1$ to guarantee that objects algebraically constructed out of g (such as $g^{\mu\nu}$, $\sqrt{\det g}$) are well controlled; (1.29) is certainly not optimal, and any other number suitable for the purposes at hand would do. To be able to prove theorems about such space-times one would need to impose some further, perhaps not necessarily uniform, decay conditions on a finite number of derivatives of g; there are various possibilities here, but we shall ignore this for the moment. Then one can define the *exterior region* $\mathscr{M}_{\mathrm{ext}}$, the *black hole region* $\mathscr{B}$ and the *future event horizon* $\mathscr{E}$ as

$$\mathscr{M}_{\mathrm{ext}} := \cup_{\tau \geq T_0} J^-(\mathcal{S}_{\tau,R_0}) = J^-(\cup_{\tau \geq T_0} \mathcal{S}_{\tau,R_0}) \;, \tag{1.30}$$

$$\mathscr{B} := \mathscr{M} \setminus \mathscr{M}_{\mathrm{ext}} \;, \quad \mathscr{E} := \partial \mathscr{B} \;, \tag{1.31}$$

where

$$\mathcal{S}_{\tau,R_0} := \{t = \tau, r = R_0\} \;. \tag{1.32}$$

We will refer to the definition (1.27)-(1.32) as that of a *naive* black hole.

In the setup of (1.27)-(1.32) an arbitrarily chosen R_0 has been used; for this definition to make sense $\mathscr{B}$ so defined should *not* depend upon this choice. This is indeed the case, as can be seen as follows:

Proposition 4 *Let $\mathscr{O}_a \subset \mathbb{R}^3 \setminus B(0, R_0)$, $a = 1, 2$, and let $\mathscr{U}_a \subset \mathscr{M}$ be of the form $\{(t \geq T_0 - R_0 + r(\boldsymbol{x}), \boldsymbol{x}) \;, \boldsymbol{x} \in \mathscr{O}_a\}$ in the coordinate system of (1.28). Then*

$$I^-(\mathscr{U}_1) = J^-(\mathscr{U}_1) = I^-(\mathscr{U}_2) = J^-(\mathscr{U}_2) \;.$$

Proof. If Γ is a future directed causal path from $p \in \mathscr{M}$ to $q = (t, \boldsymbol{x}) \in \mathscr{U}_1$, then the path obtained by concatenating Γ with the path $[0,1] \ni s \to (t(s) := t + s, \boldsymbol{x}(s) := \boldsymbol{x})$ is a causal path which is not a null geodesic, hence can be deformed to a timelike path from p to $(t+1, \boldsymbol{x}) \in \mathscr{U}_1$. It follows that $I^-(\mathscr{U}_1) = J^-(\mathscr{U}_1)$; clearly the same holds for $\mathscr{U}_2$. Next, let $\boldsymbol{x}_a \in \mathscr{O}_a$, and let $\gamma : [0,1] \to \mathbb{R}^3 \setminus B(0, R_0)$ be any differentiable path such that $\gamma(0) = \boldsymbol{x}_1$ and $\gamma(1) = \boldsymbol{x}_2$. Then for any $t_0 \geq T_0 - R_0 + r(\boldsymbol{x}_1)$ the causal curve $[0,1] \ni s \to \Gamma(s) = (t := Cs + t_0, \boldsymbol{x}(s) := \gamma(s))$ will be causal for the metric g by (1.28) if the constant C is chosen large enough, with a similar result holding when $\boldsymbol{x}_1$ is interchanged with $\boldsymbol{x}_2$. The equality $I^-(\mathscr{U}_1) = I^-(\mathscr{U}_2)$ easily follows from this observation. $\square$

Summarizing, Prop. 4 shows that there are many possible equivalent definitions of $\mathscr{M}_{\mathrm{ext}}$: in (1.30) one can replace $J^-(\mathcal{S}_{\tau,R_0})$ by $J^-(\mathcal{S}_{\tau,R_1})$ for any $R_1 \geq R_0$, but also simply by $J^-((t+\tau, q))$, for any $p = (t, q) \in \mathscr{M}$ which belongs to the region covered by the coordinate system (t, x^i).

The following remarks concerning the definition (1.30)-(1.31) are in order:

- For vacuum, stationary, asymptotically flat space-times the definition is equivalent to the usual one with $\mathscr{I}$ [46, Footnote 7, p. 572]; here the results of [49, 50] are used. However, one does not expect the existence of a smooth $\mathscr{I}^+$ to follow from (1.27)-(1.28) in general.
- Suppose that $\mathscr{M}$ admits a conformal completion in the sense defined at the beginning of Sect. 1.4, and that $\mathscr{I}$ is *semi-complete to the future*, in the sense that the Geroch-Horowitz condition of the beginning of Sect. 1.4.1 holds with $\mathscr{I} \approx \mathbb{R} \times S^2$ there replaced by $\mathscr{I} \supset \mathbb{R}^+ \times S^2$. Then for any finite interval $[T_0, T_1]$ there exists $R_0(T_0, T_1)$ and a coordinate system satisfying (1.28) and covering a set $r \geq R_0(T_0, T_1)$, $T_0 - R_0 \leq t \leq T_1 - R_0$. This follows from the Tamburino-Winicour construction of Bondi coordinates (u, r, θ, φ) near $\mathscr{I}^+$ [124], followed by the introduction of the usual Minkowskian coordinates $t = u + r$, $x = r \sin\theta \cos\varphi$, *etc.* The problem is that $R(T_1, T_2)$ could shrink to zero as T_2 goes to infinity. Thus, when $\mathscr{I}^+$ exists, conditions (1.27)-(1.28) are *uniformity* conditions on $\mathscr{I}^+$ to the future – the metric remains uniformly controlled on a uniform neighbourhood of $\mathscr{I}^+$ as the retarded time goes to infinity.
- It should not be too difficult to check whether or not the future geodesically complete space-times of Friedrich [53, 58] and of Christodoulou and Klainerman [31], or the Robinson-Trautman black-holes [32], admit coordinate systems satisfying (1.27)-(1.28).

It is not clear if asymptotically flat space-times in which no such control is available do exist at all; in fact, it is tempting to formulate the following version of the *(weak) cosmic censorship* conjecture:

> The maximal globally hyperbolic development of generic[21], asymptotically flat, vacuum initial data contains a region with coordinates satisfying (1.27)-(1.28).

Whatever the status of this conjecture, one can hardly envisage numerical simulations leading to the calculation of an essential fraction of the total energy radiated away in space-times in which some uniformity conditions do not hold.

1.8.2 Quasi-local black holes

As already argued, the naive approach of the previous section should be more convenient for numerical simulations of black hole space-times, as compared to the usual one based on Scri. It appears to be even more convenient to have a framework in which all the issues are localized in space; we wish to suggest such a framework here. When numerically modeling an asymptotically flat space-time, whether in a conformal or a direct approach, a typical numerical grid will contain large spheres S(R) on which the metric is nearly flat, so that an inequality such as (1.28)-(1.29) will hold in a neighbourhood of S(R). On slices $t = \mathrm{const}$ the condition (1.28) is usually complemented with a fall-off condition on the derivatives of the metric

$$|\partial_\sigma g_{\mu\nu}| \le C r^{-\alpha-1} , \tag{1.33}$$

however condition (1.33) is inadequate in the radiation regime, where retarded time derivatives of the metric are not expected to fall-off faster than r^{-1}. It turns out that there is a condition on derivatives of the metric in null directions which is fulfilled at large distance both in spacelike and in null directions: let K_a, $a = 1, 2$, be null future pointing vector fields on $S(R)$ orthogonal to $S(R)$, with K_1 inwards pointing and K_2 – outwards pointing; these vector fields are unique up to scaling. Let θ_a denote the associated null second fundamental forms defined as

$$\forall\ X, Y \in TS(R) \quad \chi_a(X, Y) := g(\nabla_X K_a, Y) . \tag{1.34}$$

It can be checked, e.g. using the asymptotic expansions for the connection coefficients near $\mathscr{I}^+$ from [43, Appendix C], that χ_1 is negative definite and χ_2 is positive definite for Bondi spheres $S(R)$ sufficiently close to $\mathscr{I}^+$; similarly for $\mathscr{I}^-$. This property is not affected by the rescaling freedom at hand. Following G. Galloway [60], a two-dimensional spacelike submanifold of a four-dimensional space-time will be called *weakly null convex* if χ_1 is semi-positive definite, with the trace of χ_2 negative.[22] The null convexity condition is easily verified for sufficiently large spheres in a region asymptotically flat in the sense of (1.33).

[21] The examples constructed by Christodoulou [29] with spherically symmetric gravitating scalar fields suggest that the genericity condition is unavoidable, though no corresponding vacuum examples are known.

[22] Galloway defines *null convexity* through the requirement of positive definiteness of χ_1 and negative definiteness of χ_2. However, he points out himself [60, p. 1472] that the weak null convexity as defined above suffices for his arguments to go through.

It does also hold for large spheres in a large class of space-times with negative cosmological constant. The null convexity condition is then the condition which we propose as a starting point to defining "quasi-local" black holes and horizons. The point is that several of the usual properties of black holes carry over to the weakly null convex setting. In retrospect, the situation can be summarized as follows: the usual theory of Scri based black holes exploits the existence of conjugate points on appropriate null geodesics whenever those are complete to the future; this completeness is guaranteed by the fact that the conformal factor goes to zero at the conformal boundary at an appropriate rate. Galloway's discovery in [60] is that weak null convexity of large spheres near Scri provides a second, in principle completely independent, mechanism to produce the needed focusing behaviour.

Throughout this section we will consider a globally hyperbolic space-time $(\mathscr{M}, g)$ with time function t. Let $\mathscr{T} \subset \mathscr{M}$ be a finite union of connected timelike hypersurfaces $\mathscr{T}_\alpha$ in $\mathscr{M}$, we set

$$\mathscr{S}_\tau := \{t = \tau\} , \quad \mathscr{T}(\tau) := \mathscr{T} \cap \mathscr{S}_\tau , \quad \mathscr{T}_\alpha(\tau) := \mathscr{T}_\alpha \cap \mathscr{S}_\tau . \tag{1.35}$$

For further purposes anything that happens on the exterior side of $\mathscr{T}$ is completely irrelevant, so it is convenient to think of $\mathscr{T}$ as a boundary of $\mathscr{M}$; global hyperbolicity should then be understood in the sense that $(\overline{\mathscr{M}} := \mathscr{M} \cup \mathscr{T}, g)$ is strongly causal, and that $J^+(p; \overline{\mathscr{M}}) \cap J^-(q; \overline{\mathscr{M}})$ is compact in $\overline{\mathscr{M}}$ for all $p, q \in \overline{\mathscr{M}}$. Recall that the null convergence condition is the requirement that

$$\mathrm{Ric}(X, X) \geq 0 \quad \text{for all } X \in TM . \tag{1.36}$$

We have the following *topological censorship* theorem for weakly null convex timelike boundaries:

Theorem 5 (Galloway [60]) *Suppose that a globally hyperbolic space-time $(\overline{\mathscr{M}}, g)$ satisfying the null convergence condition (1.36) has a timelike boundary $\mathscr{T} = \cup_{\alpha=1}^I \mathscr{T}_\alpha$ and a time function t such that the level sets of t are Cauchy surfaces, with each section $\mathscr{T}(\tau)$ of $\mathscr{T}$ being null convex. Then distinct $\mathscr{T}_\alpha$'s cannot communicate with each other:*

$$\alpha \neq \beta \qquad J^+(\mathscr{T}_\alpha) \cap J^-(\mathscr{T}_\beta) = \emptyset .$$

As is well known, topological censorship implies constraints on the topology:

Theorem 6 (Galloway [60]) *Under the hypotheses of Theorem 5 suppose further that the cross-sections $\mathscr{T}_\alpha(\tau)$ of $\mathscr{T}_\alpha$ have spherical topology.*[23] *Then the* α-domain of outer communication

$$\langle\langle \mathscr{T}_\alpha \rangle\rangle := J^+(\mathscr{T}_\alpha) \cap J^-(\mathscr{T}_\alpha) \tag{1.37}$$

is simply connected.

[23] The reader is referred to [61] and references therein for results without the hypothesis of spherical topology. The results there, presented in a Scri setting, generalize immediately to the weakly null convex one.

It follows in particular from Theorem 6 that $\overline{\mathscr{M}}$ can be replaced by a subset thereof such that $\mathscr{T}$ is connected in the new space-time, with all essential properties relevant for the discussion in the remainder of this section being unaffected by that replacement. We shall not do that, to avoid a lengthy discussion of which properties are relevant and which are not, but the reader should keep in mind that the hypothesis of connectedness of $\mathscr{T}$ can indeed be done without any loss of generality for most purposes.

We define the *quasi-local black hole region* $\mathscr{B}_{\mathscr{T}_\alpha}$ and the *quasi-local event horizon* $\mathscr{E}_{\mathscr{T}_\alpha}$ associated with the hypersurface $\mathscr{T}_\alpha$ by

$$\mathscr{B}_{\mathscr{T}_\alpha} := \mathscr{M} \setminus J^-(\mathscr{T}_\alpha) , \qquad \mathscr{E}_{\mathscr{T}_\alpha} := \partial \mathscr{B}_{\mathscr{T}_\alpha} . \tag{1.38}$$

If $\mathscr{T}$ is the hypersurface $\cup_{\tau \geq T_0} \mathcal{S}_{\tau, R_0}$ of Section 1.8.1 then the resulting black hole region coincides with that defined in (1.31), hence does not depend upon the choice of R_0 by Proposition 4; however, $\mathscr{B}_{\mathscr{T}_\alpha}$ might depend upon the chosen family of observers $\mathscr{T}_\alpha$ in general. It is certainly necessary to impose some further conditions on $\mathscr{T}$ to reduce this dependence. A possible condition, suggested by the geometry of the large coordinate spheres considered in the previous section, could be that the light-cones of the induced metric on $\mathscr{T}$ are uniformly controlled both from outside and inside by those of two static, future complete reference metrics on $\mathscr{T}$. However, neither the results above, nor the results that follow, do require that condition.

The Scri-equivalents of Theorem 6 [22, 46, 59, 61, 63, 84] allow one to control the topology of "good" sections of the horizon, and for the standard stationary black-holes this does lead to the usual $S^2 \times \mathbb{R}$ topology of the horizon [46, 73]. In particular, in stationary, asymptotically flat, appropriately regular space-times the intersection of a partial Cauchy hypersurface with an event horizon will necessary be a finite union of spheres. In general space-times such intersections do not even need to be manifolds: for example, in the usual spherically symmetric collapsing star the intersection of the event horizon with level sets of a time function will be a point at the time of appearance of the event horizon. We refer the reader to [38, Section 3] for other such examples, including one in which the topology of sections of horizon changes type from toroidal to spherical as time evolves. This behaviour can be traced back to the existence of past end points of the generators of the horizon. Nevertheless, some sections of the horizon have controlled topology – for instance, we have the following:

Theorem 7 *Under the hypotheses of Theorem 5, consider a connected component $\mathscr{T}_\alpha$ of $\mathscr{T}$ such that $\mathscr{E}_{\mathscr{T}_\alpha} \neq \emptyset$. Let*

$$\mathscr{C}_\alpha(\tau) := \partial J^+(\mathscr{T}_\alpha(\tau)) .$$

If $\mathscr{C}_\alpha(\tau) \cap \mathscr{E}_{\mathscr{T}_\alpha}$ is a topological manifold, then each connected component thereof has spherical topology.

Proof. Consider the open subset $\mathscr{M}_\tau$ of $\overline{\mathscr{M}}$ defined as

$$\mathscr{M}_\tau := I^+(\mathscr{C}_\alpha(\tau); \overline{\mathscr{M}}) \cap I^-(\mathscr{T}_\alpha; \overline{\mathscr{M}}) \subset \langle\langle \mathscr{T}_\alpha \rangle\rangle .$$

We claim that $(\mathscr{M}_\tau, g|_{\mathscr{M}_\tau})$ is globally hyperbolic: indeed, let $p, q \in \mathscr{M}_\tau$; global hyperbolicity of $\overline{\mathscr{M}}$ shows that $J^-(p;\overline{\mathscr{M}}) \cap J^+(q;\overline{\mathscr{M}})$ is a compact subset of $\overline{\mathscr{M}}$, which is easily seen to be included in $\mathscr{M}_\tau$. It follows that $J^-(p;\mathscr{M}_\tau) \cap J^+(q;\mathscr{M}_\tau)$ is compact, as desired. By the usual decomposition we thus have

$$\mathscr{M}_\tau \approx \mathbb{R} \times \mathcal{S} \ ,$$

where $\mathcal{S}$ is a Cauchy hypersurface for $\mathscr{M}_\tau$. Applying Theorem 6 to the globally hyperbolic space-time $\mathscr{M}_\tau$ (which has a weakly null convex boundary $\mathscr{T}_\alpha \cap \{t > \tau\}$) one finds that $\mathscr{M}_\tau$ is simply connected, and thus so is $\mathcal{S}$. Since $\mathscr{C}_\alpha(\tau)$ and $\mathscr{E}_\alpha$ are null hypersurfaces in $\mathscr{M}$, it is easily seen that the closure in $\overline{\mathscr{M}}$ of the Cauchy surface $\{0\} \times \mathcal{S}$ intersects $\mathscr{E}_\alpha$ precisely at $\mathscr{C}_\alpha(\tau) \cap \mathscr{E}_{\mathscr{T}_\alpha}$. It follows that $\mathcal{S}$ is a compact, simply connected, three dimensional topological manifold with boundary, and a classical result [74, Lemma 4.9] shows that each connected component of $\partial\mathcal{S}$ is a sphere. The result follows now from $\partial\mathcal{S} \approx \mathscr{C}_\alpha(\tau) \cap \mathscr{E}_{\mathscr{T}_\alpha}$. $\square$

Yet another class of "good sections" of $\mathscr{E}_\mathscr{T}$ can be characterized[24] as follows: suppose that $\langle\langle \mathscr{T}_\alpha \rangle\rangle \cap \mathscr{S}_\tau$ is a submanifold with boundary of $\mathscr{M}$ which is, moreover, a retract of $\langle\langle \mathscr{T}_\alpha \rangle\rangle$. Then $\langle\langle \mathscr{T}_\alpha \rangle\rangle \cap \mathscr{S}_\tau$ is simply connected by Theorem 6, and spherical topology of all boundary components of $\langle\langle \mathscr{T}_\alpha \rangle\rangle \cap \mathscr{S}_\tau$ follows again from [74, Lemma 4.9]. It is not clear whether there always exist time functions t such that the retract condition is satisfied; similarly it is not clear that there always exist τ's for which the conditions of Theorem 7 are met for metrics which are not stationary (one would actually want "a lot of τ's"). It would be of interest to understand this better.

We have an area theorem for $\mathscr{E}_\mathscr{T}$:

Theorem 8 *Under the hypotheses of Theorem 5, suppose further that $\mathscr{E}_\mathscr{T} \neq \emptyset$. Let $\mathscr{S}_a$, $a = 1, 2$ be two achronal spacelike embedded hypersurfaces of C^2 differentiability class, set $S_a = \mathscr{S}_a \cap \mathscr{E}_\mathscr{T}$. Then:*

1. *The area of S_a is well defined.*
2. *If*
$$S_1 \subset J^-(S_2) \ ,$$
then the area of S_2 is larger than or equal to that of S_1. (Moreover, this is true even if the area of S_1 is counted with multiplicity[25] of generators provided that $S_1 \cap S_2 = \emptyset$.)

We note that point 1 is less trivial as it appears, because horizons can be rather rough sets, and it requires a certain amount of work to establish that claim.

[24] I am grateful to G. Galloway for useful discussions concerning this question, as well as many other points presented in this section.
[25] See [38] for details.

Proof. The result is obtained by a mixture of methods of [38] and of [60], and proceeds by contradiction: assume that the Alexandrov divergence $\theta_{\mathcal{A}l}$ of $\mathscr{E}_{\mathscr{T}}$ is negative, and consider the $S_{\epsilon,\eta,\delta}$ deformation of the horizon as constructed in Proposition 4.1 of [38], with parameters chosen so that $\theta_{\epsilon,\eta,\delta} < 0$. Global hyperbolicity implies the existence of an achronal null geodesic from $S_{\epsilon,\eta,\delta}$ to some cut $\mathscr{T}(\tau)$ of $\mathscr{T}$. The geodesic can further be chosen to be "extremal", in the sense that it meets $\mathscr{T}(t)$ for the smallest possible value of t among all generators of the boundary of $J^+(S_{\epsilon,\eta,\delta})$ meeting $\mathscr{T}$. The argument of the proof of Theorem 1 of [60] shows that this is incompatible with the null energy condition and with weak null convexity of $\mathscr{T}(\tau)$. It follows that $\theta_{\mathcal{A}l} \geq 0$, and the result follows from [38, Proposition 3.3 and Theorem 6.1]. □

It immediately follows from the proof above that, under the hypotheses of Theorem 5, the occurrence of twice differentiable future trapped (compact) surfaces implies the presence of a black hole region. The same result holds for semi-convex compact surfaces which are trapped in an Alexandrov sense, that is, (1.21) holds with λ there defined in a way which should be clear from the discussion following Theorem 2. It is, however, not known if the existence of marginally trapped surfaces – whether defined in a classical, or Alexandrov, or a viscosity sense – does signal the occurrence of black hole; it would be of interest to settle that.

In summary, we have shown that the quasi-local black holes, defined using weakly null convex timelike hypersurfaces, or boundaries, possess several properties usually associated with the Scri-based black holes, without the associated problems. We believe they provide a reasonable alternative, well suited for numerical calculations. In the next section we address the question of numerically finding the resulting horizons.

1.8.3 Finding horizons

Consider the large coordinate spheres $\mathcal{S}_{\tau,R_1}$ of Section 1.8.1; under reasonable restrictions on $\mathscr{M}$ one would expect that $\partial J^-(\mathcal{S}_{\tau,R_1})$ converges as $\tau \to \infty$ — *e.g.* in Hausdorff topology — to the event horizon $\mathscr{E}$ of (1.31). Similarly, the achronal boundaries $\partial J^-(\mathscr{T}(\tau))$ should sometimes converge to the event horizon $\mathscr{E}_{\mathscr{T}}$. Whenever that is the case, the $\partial J^-(\mathcal{S}_{\tau,R_1})$'s or the $\partial J^-(\mathscr{T}(\tau))$'s can be taken as good approximations of the event horizon when τ is chosen to be large enough. This is of practical significance, as it provides a way of approximately locating the horizon in numerical simulations. Let us establish one such approximation result, for the $\partial J^-(\mathscr{T}(\tau))$'s; clearly the corresponding result for the $J^-(\mathcal{S}_{\tau,R_1})$'s follows by specialization. We assume that $\overline{\mathscr{M}}$ is spatially compact and globally hyperbolic, which seem rather natural assumptions in the context of numerical simulations of space-times:

Theorem 9 *Consider a globally hyperbolic space-time $(\overline{\mathscr{M}}, g)$ with compact Cauchy hypersurfaces $\mathscr{S}_\tau$ and timelike boundary $\mathscr{T}$, and suppose that*

$$\mathscr{B}_{\mathscr{T}} \neq \emptyset \,. \tag{1.39}$$

Then for any τ we have

$$\partial J^-(\mathscr{T}(\sigma)) \cap \mathscr{S}_\tau \to_{\sigma\to\infty} \mathscr{E}_{\mathscr{T}} \cap \mathscr{S}_\tau \ ,$$

where the convergence is in Hausdorff distance.

Proof. Under the usual identification of $\mathscr{M}$ with $\mathbb{R} \times \mathscr{S}_0$, let f be the graphing function of $\partial J^-(\mathscr{T}(\sigma))$ over the projection $\mathrm{pr}_2\mathscr{E}$ of $\mathscr{E}$ on $\mathscr{S}_0$; $\mathrm{pr}_2\mathscr{E}$ is an open subset of $\mathscr{S}_0$ by the invariance-of-the-domain theorem. If $p = (t, \boldsymbol{x})$ is such that $t < f(\boldsymbol{x})$, then there exists a causal curve from p to $\mathscr{T}$, hence $p \in J^-(\mathscr{T}(\sigma))$ for some σ. It follows that

$$\mathrm{pr}_2\mathscr{E} = \cup_\sigma \mathrm{pr}_2 J^-(\mathscr{T}(\sigma)) \ .$$

Let f_σ be the graphing function of $\partial J^-(\mathscr{T}(\sigma))$ over $\mathrm{pr}_2 J^-(\mathscr{T}(\sigma)) \subset \mathscr{S}_0$; clearly

$$\mathrm{pr}_2 J^-(\mathscr{T}(\sigma)) \subset \mathrm{pr}_2 J^-(\mathscr{T}(\sigma')) \qquad \sigma < \sigma' \ ,$$

which shows that f and all the $f_{\sigma'}$'s are defined on $\mathrm{pr}_2 J^-(\mathscr{T}(\sigma))$ for $\sigma' > \sigma$. The f_τ's are uniformly Lipschitz and bounded over any compact subset $K \subset \mathrm{pr}_2\mathscr{E}$ of $\mathscr{S}_0$, and hence for any sequence τ_i there exists a subsequence such that the $f_{\tau_{i_j}}$'s converge to a Lipschitz function h. Since $f_\tau \le f$ for all τ we have $h \le f|_K$. Suppose that there exists $\boldsymbol{x}$ such that $h(\boldsymbol{x}) < f(\boldsymbol{x})$, then $p = ((h(\boldsymbol{x}) + f(\boldsymbol{x}))/2, \boldsymbol{x})$ is in $J^-(\mathscr{T})$, hence in $J^-(\mathscr{T}(\sigma))$ for some σ, so that $(h(\boldsymbol{x}) + f(\boldsymbol{x}))/2 \le f_\sigma(\boldsymbol{x})$ which is not possible; we thus have $h = f|_K$. It follows that the f_τ's converge pointwise to f. The f_τ's are monotonously increasing as τ increases, and Dini's theorem implies that the f_τ's converge uniformly to f over any compact set. The result follows now by elementary considerations using the fact that $\mathscr{E}_{\mathscr{T}} \cap \mathscr{S}_\tau$ is compact, being the intersection of a closed set with a compact one. □

Theorem 9 gives partial justification for the numerical analysis of Aninos [7], Libson et al. [98], or Massó et al. [102],[26] where the event horizon $\mathscr{E}$ is taken to be

$$\mathscr{E} \approx \partial J^-(\mathcal{S}_{\tau,R_1}) \ , \quad \tau \text{ as big as the computer simulation allows} \ .$$

In those works $\partial J^-(\mathcal{S}_{\tau,R_1})$ is further numerically approximated as the solution of an eikonal equation, which leads to difficulties at past end points of the generators of $\partial J^-(\mathcal{S}_{\tau,R_1})$. We conjecture that a more stable method of locating objects such as $\partial J^-(\mathcal{S}_{\tau,R_1})$ is provided by the following straightforward modification of the Fermat-type variational principle presented in the Appendix below, see (1.43): let $\mathcal{P}(x)$ be defined as in the paragraph preceding (1.43), with $\mathscr{H}_\sigma$ there replaced by $\mathcal{S}_{\tau,R_1}$, and let $\hat\gamma$ be the null lifts of paths γ as defined in the

[26] We note that the claim in [7, 98], that the horizon acts as an attractor for backwards-directed null geodesics, does not seem to be justified. In fact, such a statement is coordinate-dependent: it is easy to introduce coordinate systems in the Schwarzschild space-time where backwards-directed null geodesics will be actually repelled by the horizon.

paragraph preceding (1.45). An argument similar to that of the proof of Corollary A.3 shows that in globally hyperbolic space-times the set $\partial J^-(\mathcal{S}_{\tau,R_1})$ is a graph of a function f such that

$$f(\boldsymbol{x}) = \sup_{\hat\gamma\in\mathcal{P}(x)} t(\hat\gamma(a)) \ . \tag{1.40}$$

Such a maximization over null lifts $\hat\gamma$ of paths γ which have image in a fixed spacelike hypersurface should be easy to perform numerically by e.g. Newton's method. The variational principle above automatically takes care of the past end-points of the horizon.

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Appendix A: Future horizons are semi-convex

A hypersurface $\mathscr{H} \subset M$ will be said to be *future null geodesically ruled* if every point $p \in \mathscr{H}$ belongs to a future inextensible null geodesic $\Gamma \subset \mathscr{H}$; those geodesics will be called *the generators* of $\mathscr{H}$. We emphasize that the generators are allowed to have past endpoints on $\mathscr{H}$, but no future endpoints. *Past null geodesically ruled* hypersurfaces are defined by changing the time orientation. Examples of future geodesically ruled hypersurfaces include past Cauchy horizons $\mathcal{D}^-(\Sigma)$ of achronal sets Σ [115, Theorem 5.12] and black hole event horizons $\partial J^-(\mathscr{I}^+)$ [73, p. 312].

Let $\dim M = n+1$ and suppose that $\mathcal{O}$ is a domain in $\mathbb{R}^n$. Recall that a continuous function $f : \mathcal{O} \to \mathbb{R}$ is called semi–convex if there exists a C^2 function $\phi : \mathcal{O} \to \mathbb{R}$ such that $f + \phi$ is convex. We shall say that the graph of f is a semi–convex hypersurface if f is semi–convex. A hypersurface $\mathscr{H}$ in a manifold M will be said semi–convex if $\mathscr{H}$ can be covered by coordinate patches $\mathcal{U}_\alpha$ such that $\mathscr{H} \cap \mathcal{U}_\alpha$ is a semi–convex graph for each α.

Consider an achronal hypersurface $\mathscr{H} \neq \emptyset$ in a globally hyperbolic space–time (M, g). Let t be a time function on M which induces a diffeomorphism of M with $\mathbb{R}\times\Sigma$ in the standard way [64, 122], with the level sets $\Sigma_\tau \equiv \{p | t(p) = \tau\}$ of t being Cauchy surfaces. As usual we identify Σ_0 with Σ, and in the identification above the curves $\mathbb{R} \times \{q\}$, $q \in \Sigma$, are integral curves of ∇t. Define

$$\Sigma_{\mathscr{H}} = \{q \in \Sigma | \mathbb{R} \times \{q\} \text{ intersects } \mathscr{H}\} \ . \tag{1.41}$$

For $q \in \Sigma_{\mathscr{H}}$ the set $(I \times q) \cap H$ is a point by achronality of $\mathscr{H}$, we shall denote this point by $(f(q), q)$. Thus an achronal hypersurface $\mathscr{H}$ in a globally hyperbolic space–time is a graph over $\Sigma_{\mathscr{H}}$ of a function f. The invariance-of-the-domain theorem shows that $\Sigma_{\mathscr{H}}$ is an open subset of Σ. We have the following:

Theorem A.1 *Let $\mathscr{H} \neq \emptyset$ be an achronal future null geodesically ruled hypersurface in a globally hyperbolic space–time $(M = \mathbb{R}\times\Sigma, g)$. Then $\mathscr{H}$ is the graph of a semi–convex function f defined on an open subset $\Sigma_{\mathscr{H}}$ of Σ, in particular $\mathscr{H}$ is semi–convex.*

Proof. As discussed above, $\mathscr{H}$ is the graph of a function f. The idea of the proof is to show that f satisfies a variational principle, the semi–concavity of f follows then by a standard argument. Let $p \in \mathscr{H}$ and let $\mathcal{O}$ be a coordinate patch in a neighborhood of p such that $x^0 = t$, with $\mathcal{O}$ of the form $I \times B(3R)$, where $B(R)$ denotes a coordinate ball centered at 0 of radius R in $\mathbb{R}^3$, with $p = (t(p), 0)$. Here I is the range of the coordinate x^0, we require it to be a bounded interval the size of which will be determined later on. We further assume that the curves $I \times \{\boldsymbol{x}\}$, $\boldsymbol{x} \in B(3R)$, are integral curves of ∇t. Define

$$\mathcal{U}_0 = \{\boldsymbol{x} \in B(3R) |\text{ the causal path } I \ni t \to (t, \boldsymbol{x}) \text{ intersects } \mathscr{H}\} \ .$$

We note that $\mathcal{U}_0$ is non–empty, since $0 \in \mathcal{U}_0$. Set

$$\mathscr{H}_\sigma = \mathscr{H} \cap \Sigma_\sigma \ , \tag{1.42}$$

and choose σ large enough so that $\mathcal{O} \subset I^-(\Sigma_\sigma)$. Now p lies on a future inextensible generator Γ of $\mathscr{H}$, and global hyperbolicity of (M, g) implies that $\Gamma \cap \Sigma_\sigma$ is nonempty, hence $\mathscr{H}_\sigma$ is nonempty.

For $\boldsymbol{x} \in B(3R)$ let $\mathcal{P}(x)$ denote the collection of piecewise differentiable future directed null curves $\Gamma : [a, b] \to M$ with $\Gamma(a) \in \mathbb{R} \times \{\boldsymbol{x}\}$ and $\Gamma(b) \in \mathscr{H}_\sigma$. We define

$$\tau(\boldsymbol{x}) = \sup_{\Gamma \in \mathcal{P}(x)} t(\Gamma(a)) \ . \tag{1.43}$$

We emphasize that we allow the domain of definition $[a, b]$ to depend upon Γ, and that the "a" occurring in $t(\Gamma(a))$ in (1.43) is the lower bound for the domain of definition of the curve Γ under consideration. We have the following result (compare [8, 68, 116]):

Proposition A.2 (Fermat principle) *For $\boldsymbol{x} \in \mathcal{U}_0$ we have*

$$\tau(\boldsymbol{x}) = f(\boldsymbol{x}) \ .$$

Proof. Let Γ be any generator of $\mathscr{H}$ such that $\Gamma(0) = (f(\boldsymbol{x}), \boldsymbol{x})$, clearly $\Gamma \in \mathcal{P}(x)$ so that $\tau(\boldsymbol{x}) \geq f(\boldsymbol{x})$. To show that this inequality has to be an equality, suppose for contradiction that $\tau(\boldsymbol{x}) > f(\boldsymbol{x})$, thus there exists a null future directed curve Γ such that $t(\Gamma(0)) > f(\boldsymbol{x})$ and $\Gamma(1) \in \mathscr{H}_\sigma \subset \mathscr{H}$. Then the curve $\tilde{\Gamma}$ obtained by following $\mathbb{R} \times \{\boldsymbol{x}\}$ from $(f(\boldsymbol{x}), \boldsymbol{x})$ to $(t(\Gamma(0)), \boldsymbol{x})$ and following Γ from there on is a causal curve with endpoints on $\mathscr{H}$ which is not a null geodesic. By [73, Proposition 4.5.10] $\tilde{\Gamma}$ can be deformed to a timelike curve with the same endpoints, which is impossible by achronality of $\mathscr{H}$. □

The Fermat principle, Proposition A.2, shows that f is a solution of the variational principle (1.43). Now this variational principle can be rewritten in a somewhat more convenient form as follows: The identification of M with $\mathbb{R}\times\Sigma$ by flowing from $\Sigma_0\equiv\Sigma$ along the gradient of t leads to a global decomposition of the metric of the form

$$g=\alpha(-dt^2+h_t)\ ,$$

where h_t denotes a t–dependent family of Riemannian metrics on Σ. Any future directed differentiable null curve $\Gamma(s)=(t(s),\gamma(s))$ satisfies

$$\frac{dt(s)}{ds}=\sqrt{h_{t(s)}(\dot\gamma,\dot\gamma)}\ ,$$

where $\dot\gamma$ is a shorthand for $d\gamma(s)/ds$. It follows that for any $\Gamma\in\mathcal{P}(x)$ it holds that

$$\begin{aligned} t(\Gamma(a)) &= t(\Gamma(b))-\int_a^b\frac{dt}{ds}ds \\ &= \sigma-\int_a^b\sqrt{h_{t(s)}(\dot\gamma,\dot\gamma)}ds\ . \end{aligned}$$

This allows us to rewrite (1.43) as

$$\tau(\boldsymbol{x})=\sigma-\mu(\boldsymbol{x})\ ,\qquad \mu(\boldsymbol{x})\equiv\inf_{\Gamma\in\mathcal{P}(x)}\int_a^b\sqrt{h_{t(s)}(\dot\gamma,\dot\gamma)}ds\ . \tag{1.44}$$

We note that in static space–times $\mu(\boldsymbol{x})$ is the Riemannian distance from $\boldsymbol{x}$ to $\mathscr{H}_\sigma$. In particular Equation (1.44) implies the well known fact, that in globally hyperbolic static space–times Cauchy horizons of open subsets of level sets of t are graphs of the distance function from the boundary of those sets.

Let $\gamma:[a,b]\to\Sigma$ be a piecewise differentiable path, for any $p\in\mathbb{R}\times\{\gamma(b)\}$ we can find a null future directed curve $\hat\gamma:[a,b]\to M$ of the form $\hat\gamma(s)=(\phi(s),\gamma(s))$ with future end point p by solving the problem

$$\begin{cases}\phi(b)=t(p)\ ,\\ \dfrac{d\phi(s)}{ds}=\sqrt{h_{\phi(s)}(\dot\gamma(s),\dot\gamma(s))}\ .\end{cases} \tag{1.45}$$

The path $\hat\gamma$ will be called the *null lift of* γ *with endpoint* p.

As an example of application of Proposition A.2 we recover the following well known result [115]:

Corollary A.3 *f is Lipschitz continuous on any compact subset of its domain of definition.*

Proof. For $\boldsymbol{y},\boldsymbol{z}\in B(2R)$ let $K\subset\mathbb{R}\times B(2R)$ be a compact set which contains all the null lifts $\Gamma_{\boldsymbol{y},\boldsymbol{z}}$ of the coordinate segments $[\boldsymbol{y},\boldsymbol{z}]:=\{\lambda\boldsymbol{y}+(1-\lambda)\boldsymbol{z}\ ,\ \lambda\in[0,1]\}$ with endpoints $(\tau(\boldsymbol{z}),\boldsymbol{z})$. Define

$$\hat C=\sup\{\sqrt{h_p(n,n)}|p\in K,|n|_\delta=1\}\ , \tag{1.46}$$

where the supremum is taken over all points $p \in K$ and over all vectors $n \in T_pM$ the coordinate components n^i of which have Euclidean length $|n|_\delta$ equal to one. Choose I to be a bounded interval large enough so that $K \subset I \times B(2R)$ and, as before, choose σ large enough so that $I \times B(2R)$ lies to the past of Σ_σ. Let $\boldsymbol{y}, \boldsymbol{z} \in B(2R)$ and consider the causal curve $\Gamma = (t(s), \gamma(s))$ obtained by following the null lift $\Gamma_{\boldsymbol{y},\boldsymbol{z}}$ in the parameter interval $s \in [0,1]$, and then a generator of $\mathscr{H}$ from $(\tau(\boldsymbol{z}), \boldsymbol{z})$ until $\mathscr{H}_\sigma$ in the parameter interval $s \in [1,2]$. Then we have

$$\mu(\boldsymbol{z}) = \int_1^2 \sqrt{h_{t(s)}(\dot\gamma, \dot\Gamma)} ds \ .$$

Further $\Gamma \in \mathcal{P}(x)$ so that

$$\begin{aligned} \mu(\boldsymbol{y}) &\le \int_0^2 \sqrt{h_{t(s)}(\dot\gamma, \dot\gamma)} ds \\ &= \int_0^1 \sqrt{h_{t(s)}(\dot\gamma, \dot\gamma)} ds + \int_1^2 \sqrt{h_{t(s)}(\dot\gamma, \dot\gamma)} ds \\ &\le \hat{C} |\boldsymbol{y} - \boldsymbol{z}|_\delta + \mu(\boldsymbol{z}) \ , \end{aligned} \tag{1.47}$$

where $|\cdot|_\delta$ denotes the Euclidean norm of a vector, and with $\hat{C}$ defined in (1.46). Setting 1) first $\boldsymbol{y} = \boldsymbol{x}$, $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{h}$ in (1.47) and 2) then $\boldsymbol{z} = \boldsymbol{x}$, $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{h}$, the Lipschitz continuity of f on $B(2R)$ follows. The general result is obtained now by a standard covering argument. □

Returning to the proof of Theorem A.1, for $\boldsymbol{x} \in B(R)$ let $\Gamma_{\boldsymbol{x}}$ be a generator of $\mathscr{H}$ such that $\Gamma_{\boldsymbol{x}}(0) = (\tau(\boldsymbol{x}), \boldsymbol{x})$, and, if we write $\Gamma_{\boldsymbol{x}}(s) = (\phi_{\boldsymbol{x}}(s), \gamma_{\boldsymbol{x}}(s))$, then we require that $\gamma_{\boldsymbol{x}}(s) \in B(2R)$ for $s \in [0,1]$. For $s \in [0,1]$ and $\boldsymbol{h} \in B(R)$ let $\gamma_{\boldsymbol{x},\pm}(s) \in \Sigma$ be defined by

$$\gamma_{\boldsymbol{x},\pm}(s) = \gamma_{\boldsymbol{x}}(s) \pm (1-s)\boldsymbol{h} = s\gamma_{\boldsymbol{x}}(s) + (1-s)(\gamma_{\boldsymbol{x}}(s) \pm \boldsymbol{h}) \in B(2R) \ .$$

We note that

$$\gamma_{\boldsymbol{x},\pm}(0) = \boldsymbol{x} \pm \boldsymbol{h} \ , \qquad \gamma_{\boldsymbol{x},\pm}(1) = \gamma_{\boldsymbol{x}}(1) \ , \qquad \dot\gamma_{\boldsymbol{x},\pm} - \dot\gamma_{\boldsymbol{x}} = \mp \boldsymbol{h} \ .$$

Let $\Gamma_{\boldsymbol{x},\pm} = (\phi_\pm(s), \gamma_{\boldsymbol{x},\pm})$ be the null lifts of the paths $\gamma_{\boldsymbol{x},\pm}$ with endpoints $\Gamma_{\boldsymbol{x}}(1)$. Let K be a compact set containing all the $\Gamma_{\boldsymbol{x},\pm}$'s, where $\boldsymbol{x}$ and $\boldsymbol{h}$ run through $B(R)$. Let I be any bounded interval such that $I \times B(2R)$ contains K. As before, choose σ so that $I \times B(2R)$ lies to the past of Σ_σ, and let b be such that $\Gamma_{\boldsymbol{x}}(b) \in \mathscr{H}_\sigma$. (The value of the parameter b will of course depend upon $\boldsymbol{x}$). Let $\Gamma_\pm$ be the null curve obtained by following $\Gamma_{\boldsymbol{x},\pm}$ for parameter values $s \in [0,1]$, and then $\Gamma_{\boldsymbol{x}}$ for parameter values $s \in [1,b]$. Then $\Gamma_\pm \in \mathcal{P}(\boldsymbol{x} \pm \boldsymbol{h})$ so that we have

$$\mu(\boldsymbol{x} \pm \boldsymbol{h}) \le \int_0^1 \sqrt{h_{\phi_\pm(s)}(\dot\gamma_{\boldsymbol{x},\pm}, \dot\gamma_{\boldsymbol{x},\pm})} ds + \int_1^b \sqrt{h_{\phi_\pm(s)}(\dot\gamma_{\boldsymbol{x}}, \dot\gamma_{\boldsymbol{x}})} ds \ .$$

Further

$$\mu(\boldsymbol{x}) = \int_0^b \sqrt{h_{\phi_{\boldsymbol{x}}(s)}(\dot\gamma_{\boldsymbol{x}},\dot\gamma_{\boldsymbol{x}})}ds \;,$$

hence

$$\begin{aligned}&\frac{\mu(\boldsymbol{x}+\boldsymbol{h})+\mu(\boldsymbol{x}-\boldsymbol{h})}{2}-\mu(\boldsymbol{x})\le\\ &\int_0^1\left(\frac{\sqrt{h_{\phi_\pm(s)}(\dot\gamma_{\boldsymbol{x},+},\dot\gamma_{\boldsymbol{x},+})}+\sqrt{h_{\phi_\pm(s)}(\dot\gamma_{\boldsymbol{x},-},\dot\gamma_{\boldsymbol{x},-})}}{2}-\sqrt{h_{\phi_{\boldsymbol{x}}(s)}(\dot\gamma_{\boldsymbol{x}},\dot\gamma_{\boldsymbol{x}})}\right)ds\;.\end{aligned} \tag{1.48}$$

Since solutions of ODE's with parameters are differentiable functions of those, we can write

$$\phi_\pm(s)=\phi_{\boldsymbol{x}}(s)+\psi_i(s)h^i+r(s,h),\qquad |r(s,h)|\le C|h|^2_\delta\;, \tag{1.49}$$

for some functions ψ_i, with a constant C which is independent of $\boldsymbol{x},\boldsymbol{h}\in B(R)$ and $s\in[0,1]$. Inserting (1.49) in (1.48), second order Taylor expanding the function $\sqrt{h_{\phi_\pm(s)}(\dot\gamma_{\boldsymbol{x},\pm},\dot\gamma_{\boldsymbol{x},\pm})}(s)$ in all its arguments around $(\phi_{\boldsymbol{x}}(s),\gamma_{\boldsymbol{x}}(s),\dot\gamma_{\boldsymbol{x}}(s))$ and using compactness of K one obtains

$$\frac{\mu(\boldsymbol{x}+\boldsymbol{h})+\mu(\boldsymbol{x}-\boldsymbol{h})}{2}-\mu(\boldsymbol{x})\le C|\boldsymbol{h}|^2_\delta\;, \tag{1.50}$$

for some constant C. Set

$$\psi(\boldsymbol{x})=\mu(\boldsymbol{x})-C|\boldsymbol{x}|^2_\delta\;.$$

Equation (1.50) shows that

$$\forall \boldsymbol{x},\boldsymbol{h}\in B(R)\qquad \psi(\boldsymbol{x})\ge\frac{\psi(\boldsymbol{x}+\boldsymbol{h})+\psi(\boldsymbol{x}-\boldsymbol{h})}{2}\;.$$

A standard argument implies that ψ is concave. It follows that

$$f(\boldsymbol{x})+C|\boldsymbol{x}|^2_\delta=\tau(\boldsymbol{x})+C|\boldsymbol{x}|^2_\delta=\sigma-\mu(\boldsymbol{x})+C|\boldsymbol{x}|^2_\delta=\sigma-\psi(\boldsymbol{x})$$

is convex, which is what had to be established. □

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